Curriculum Inspirations

Inspiring students with rich content from the MAA American Mathematics Competitions

MAA

Curriculum Burst 93: Probably Above a Parabola

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> Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a,b) lies above the parabola $y = ax^2 - bx$

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QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the upper high-school grade levels.

MATHEMATICAL TOPICS

Graphs of Functions; Probability

COMMON CORE STATE STANDARDS

A-REI.D Represent and solve equations and inequalities graphically.

S-CP.B Use the rules of probability to compute probabilities of compound events in a uniform probability model

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 7: **PERSEVERANCE IS KEY**

SOURCE: This is question # 14 from the 2011 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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This feels like a strange question. Just make sure I understand it: We pick a number a at random, which could be 1, 2, 3, …, 9, another single-digit number b from the same set, use them to draw a parabola:

$$y = ax^2 - bx$$

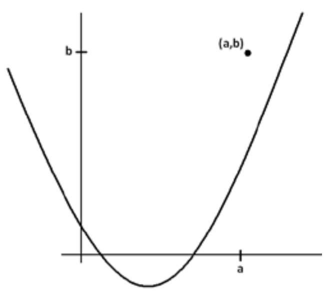
(so we get something like $y = 3x^2 - 9x$, or $y = x^2 - 4x$, always upward-facing U-shapes), and we want the point (a,b) in the picture to be above the curve.

Is $(3,9)$ above $y = 3x^2 - 9x$?

Is $(1,4)$ above $y = x^2 - 4x$?

How can I tell?

We could graph all the possibilities and check by hand.



There are nine possible values for a and nine for b. That's only 81 pictures to draw and check. (Yeesh!)

Algebraically, what does it mean for the point (a,b) to be above the curve? We need b to be larger than the y-coordinate of the point on the curve with the same x-coordinate as (a,b).

This is weird. The y-coordinate for $y = ax^2 - bx$ with the matching x-coordinate $x = a$ is $y = a \cdot a^2 - ba$. (Too many a s and b s!)

So we want: $b > a^3 - ab$. What are the chances of this happening?

Well, perhaps we can go through cases for this.

If $a = 1$, then we need $b > 1 - b$, that is, $2b > 1$ or $b > 1/2$. Any of the nine values 1, 2, 3, …, 9 for b work.

If $a = 2$, then we need $b > 8 - 2b$, that is, $b > 8/3$. Here $b = 3,4,5,6,7,8,9$ work, seven possibilities.

If $a = 3$, then we need $b > 27 - 3b$, that is, $b > 6\frac{3}{4}$.

There are three values for b that work.

If $a = 4$, then we need $b > 64 - 4b$, that is, $b > 12\frac{4}{5}$.

None work.

I suspect the numbers just keep getting bigger and the nine plus seven plus three options we have seen are it. To check, let's solve for b in general:

$$b > a^3 - ab$$
$$(a+1)b > a^3$$
$$b > \frac{a^3}{a+1}$$

Hmm. Is this always bigger than 9 when a is bigger than 4? For $a = 5,6,7,8,9$ we get $\frac{a^3}{a+1}$ equal to $\frac{125}{6}, \frac{216}{7}, \frac{343}{8}, \frac{512}{9}, \frac{729}{10}$, in turn, and each numerator is more than ten times the denominator.

Okay, we have $9 + 7 + 3 = 19$ pairs (a,b) that work, out of 81 possible pairs. The probability we seek is thus $\frac{19}{81}$.

Extension: Edna wrote $\frac{a^3}{a+1} = \frac{a^3 + a^2 - a^2}{a+1} = a^2 - \frac{a^2}{a+1}$ and then $\frac{a^2}{a+1} = \frac{a^2 + a - a}{a+1} = a - \frac{a}{a+1}$. She proved that $\frac{a^3}{a+1}$ is larger than $(a-1)^2$ for all positive integers. How? Is this true for negative integers too? For all real numbers?

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