Curriculum Burst 94: Hundreds Digit of a Power

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What is the hundreds digit of 2011^{2011} ?

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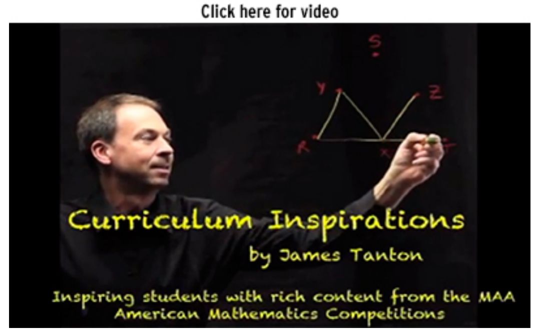

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grade levels.

MATHEMATICAL TOPICS

The binomial theorem.

COMMON CORE STATE STANDARDS

A-APR.C5 (+) Know and apply the Binomial Theorem for the expansion of (x+y)^n in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING**

SOURCE: This is question # 23 from the 2011 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I have mixed feelings about this problem. It is comprehensible – I completely understand what the question is and what I am looking for – but at the same time I don't have a clue how to go about what I am being asked to do! 2011^{2011} is some huge number. I don't want to work out what that number is.

Okay. Deep breath. All I need is the hundreds digit of 2011^{2011}.

Hmm. What does that mean? What is the hundreds digit of a number?

Consider 348712, for instance. The third-to-last digit, 7, is the hundreds digit. It is the multiple of 100 we need to write it in base ten.

$$348712 =$$
$$3\times 100000 + 4\times 10000 + 8\times 1000 + \underline{7\times 100} + 1\times 10 + 2\times 1$$

So all I need to do is to work out how many multiples of 100 there are in 2011^{2011}.

Hmm. Easier said than done.

So we need to focus on the multiples of 100. Hmm.

Can I work out what 2011^{2011}, at least as far to see how it reads as $something + m\times 100 + n\times 10 + 1$. (I can see that 2011^{2011} ends in a one! Can you also see why too?) This question wants the value of m.

Well, $2011^{2011} = (2000+10+1)^{2011}$. When I expand this out there will be lots of $2000s$ and lots of $10s$ that get multiplied together. Actually, if I am focused only on the multiples of 100 I can ignore all the products that involve the number 2000. Surely that means something?

All this reminds me of the binomial theorem. (Actually, it's the trinomial theorem):

$$(x+y+z)^N = sum\ of\ terms\ \frac{N!}{a!b!c!}x^a y^b z^c$$

where $a+b+c=N$. (See lesson 3.5 of http://gdaymath.com/courses/permutations-and-combinations/)

Here $x=2000$, $y=10$, and $z=1$, and we can ignore all the terms that involve x. That is, we need only look the terms with $a=0$. We also don't care about y^b with $b>2$ as that gets me into the thousands as well. So $a=0$ and $b=0,1,2$ is all I need write out. Okay then!

$$\begin{aligned}(2000+10+1)^{2011} &= thousands\ stuff + \frac{2011!}{0!2!2009!}2000^0 10^2 1^{2009} \\ &\quad + \frac{2011!}{0!1!2010!}2000^0 10^1 1^{2010} + \frac{2011!}{0!0!2011!}2000^0 10^0 1^{2011}\end{aligned}$$

$$= thousands\ stuff + 2011\times 1005\times 100 + 2011\times 10 + 1$$

$$= thousands\ stff\ + (2011000+10055)\times 100 + 20110 + 1$$

$$= thousands\ stuff\ + thousands\ stuff\ + 1005500 + 20111$$

So 2011^{2011} ends with 611 and has hundreds digit 6!

Extension: What is the hundreds digit of 2111^{2111}?

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