Curriculum Burst 139: Perfect Square Fraction

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> For how many integers n is $\dfrac{n}{20-n}$ the square of an integer?

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Algebra: Identify parts of equations in context

COMMON CORE STATE STANDARDS

A-SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 10: GO TO EXTREMES

SOURCE: This is question # 16 from the 2002 MAA AMC 10B Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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This question is a bit odd. We want a fraction to be the square of an integer?

To get a feel for it, let me try some values of n and see what sort of fractions we're talking about.

$n=1$ gives $\frac{1}{20-1}=\frac{1}{19}$, not even an integer.

$n=2$ gives $\frac{2}{18}=\frac{1}{9}$, not even an integer.

How about something on the extreme end of this?

$n=0$ gives $\frac{0}{20}=0$. That's a perfect square!

How about lower still, say, n a negative integer?

$n=-a$ (with a positive) gives

$$\frac{-a}{20+a}=-\frac{a}{20+a}.$$

This s a negative quantity and so won't be a square.

How about the other extreme?

$n=1000000$ gives $\frac{1000000}{-999980}$, a negative number.

Okay, so I see now that n has to be between 0 and 20, and in such a way that makes $\frac{n}{20-n}$ an integer. How about $n=10$?

$n=10$ gives $\frac{10}{10}=1$, a perfect square.

Either side of this:

$n=9$ gives $\frac{9}{11}$ a fraction smaller than 1

$n=11$ gives $\frac{11}{9}$, a fraction larger than 1.

Hmm. I can see that if $n<10$, then $\frac{n}{20-n}$ will be a fraction smaller than 1. So we need:

$$10<n<20.$$

(We can see that $n=20$ is no good for his formula!)

There are only a few remaining values we haven't yet tried:

$n=12$ gives $\frac{12}{8}=\frac{3}{2}$. $n=13$ gives $\frac{13}{7}$.

$n=14$ gives $\frac{14}{6}$. $n=15$ gives $\frac{15}{5}=3$.

$n=16$ gives $\frac{16}{4}=4=2^2$. $n=17$ gives $\frac{17}{3}$.

$n=18$ gives $\frac{18}{2}=9=3^2$. $n=19$ gives $\frac{19}{1}=19$.

Okay. We get a perfect square for $n=0,10,16,$ and 18. That's for FOUR values of n.

Extension: The 3×6 rectangle has the property that its area and its perimeter have the same numerical value. ($3\times 6=3+3+6+6$) Find the dimensions of all other rectangles with integer sides possessing this property – and prove that your list is complete!

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