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TANTON'S TAKE ON …

MODELS OF MATHEMATICS

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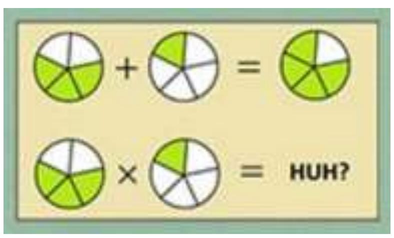
So much of the mathematics curriculum, the early curriculum in particular, constructs models to help explain to children how numbers work. For example, in the world of counting numbers, multiplication is repeated addition (3×4 is rightfully interpreted there as "three groups of four") and fractions are not so much numbers but the results of actions on different wholes: half of a pie, one sixth of the kittens, one-and-a-half times as much money.

And this is fine and dandy. But often the curriculum forgets to point out that models are being used only to motivate beginning ideas and to illustrate facets of our intuition that feel like "truth." No model can speak to what we might later decide to be the whole truth. For instance, in "groups of" thinking am I justified in saying that $0 \times 4 = 3$? After all, there are no groups of four among three objects. Is it possible to add these non-numbers called fractions? What is one third of the kittens plus half of a pie?

A typical curriculum will subtly adjust and tweak the details of the model as it goes along, both to extend its power and to overcome philosophical misgivings. Since "half of a group of ten" makes intuitive sense, let's say that $\frac{1}{2} \times 10 = 5$ and thus extend our notion of multiplication to fractions. (So we are simply declaring that "of" means "multiply" it seems.) We might also insist that the addition of fractions be done only if those fractions refer to the same whole: half a pie and a third of a pie, clearly gives five-sixths of a pie and so justifies writing $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$.

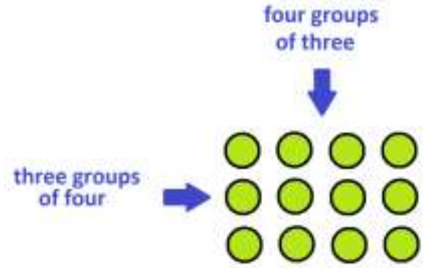
But matters inevitably become hazy as we push the models we cling to further. For example, I think I can make sense of $2\times(-3)$, two groups of negative three, but $(-2)\times 3$ has me flummoxed: I have no idea what negative two groups of something are.[1]

Also, I see how I can add two pieces of pie. But I have no idea what it means to multiply pieces of pie. What does it mean to multiply fractions, even if they are parts of the same whole?[2]



My point is that every model we devise has limitations and will "break down" at some point. In the end, it is up to us to settle on some key properties of numbers, the ones that somehow feel universal to us, and just accept those rules as beliefs. (And our next job is to detail the logical consequences of those choices.)

Don't get me wrong, models are good for exploring the properties of different types of numbers to help us decide on possible universal beliefs about them. For example, this picture shows that three groups of four really is the same as four groups of three.



Moreover, it leads the way to explaining why, in our "groups of" model, $a\times b$ is sure to have the same value of $b\times a$ no matter the counting numbers a and b. The commutativity property for multiplication is a truth in this view of counting number arithmetic. It is a very appealing truth. It feels like a fundamental truth, so fundamental that we feel compelled to believe that $a\times b=b\times a$ should hold for <u>all</u> types of numbers a and b, even though we don't know how to justify this so for negative numbers, fractions, and irrational numbers.

I have no objection to making choices like these. (Welcome to the art of doing mathematics!) So let's go ahead and choose to believe that multiplication is commutative for all numbers. But let's not hide from our students the fact that we've just made a choice to believe.

This means we must be explicit with our students about the limitations of models

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[1] I do know that the "opposite" of two groups of three would be negative six. (Opposites: Another model!) But is it obvious that $-(2\times 3)$ is the same as $(-2)\times 3$? They read differently. Or maybe, in an attempt to salvage the model, we should just declare that $(-2)\times 3$ is the same as $3\times(-2)$, which is three groups of negative two. (Can we do this? Should we believe the commutativity property holds even for negative numbers? What makes us believe in the commutative property in the first place?)

[2] I have considerable difficulty with the changing models of fractions throughout the typical curriculum.
See http://gdaymath.com/courses/fractions-are-hard/.

and point how, and when, each model eventually breaks down. (I understand that this is sophisticated conversation, best reserved for upper-middle school and high school students.) Let's be clear about which properties of numbers highlighted by a model we are simply choosing to believe extend beyond the limitations of that model.[3]

An Example: The Power and the Limitations of a Model

Some texts define 2^n, for n is a positive whole number greater than one, as "the number two multiplied by itself n times." For example,

$$2^3 = 2 \times 2 \times 2 = 8$$

and

$$2^{10} = \overbrace{2 \times 2 \times \cdots \times 2}^{\text{ten times}} = 1024\,.$$

Question: What is 2^1 here, the result of multiplying 2 by itself one time? (Is the answer four?)

This definition is certainly challenged if we extend matters to other types of exponents. What is 2^0, the result of multiplying two by itself zero times? (Does that just leave the number as two?) What is 2^{-1}, the result of multiplying two by itself negative one times? What is 2^{-2}? And so on.

But here is a lovely model to the rescue!

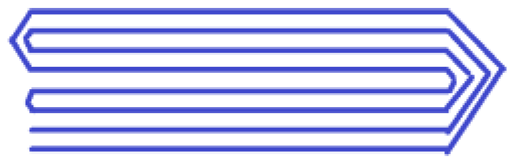
Take a strip of paper and fold it in half one time. That makes two layers of paper.

1 fold $\leftrightarrow$ 2 layers.

Fold that strip in half a second time and the count of layers doubles.

2 folds $\leftrightarrow$ $2 \times 2 = 2^2 = 4$ layers.

A third fold doubles the number of layers again.



3 folds $\leftrightarrow$ $2 \times 2 \times 2 = 2^3 = 8$ layers.

In general, for n folds we get 2^n layers, at least for n a positive counting number greater than one.

This model allows us to make sense of 2^1. In fact, we've already seen that one fold gives two layers. So according to this model we have

$$2^1 = 2\,.$$

[3] We have such conversations in the study of geometry. (What properties of parallel lines do we choose to believe? Which properties of rigid motions shall we accept? And so on.) So why not in the study of arithmetic and algebra?

The answer is that we never entertain alternative systems of arithmetic in the K-12 curriculum and so we cannot conceive of any different type of arithmetic. All properties of numbers feel like pre-ordained edicts, and the matter of making belief choices is moot. In geometry, at least, we are vaguely aware of alternative systems of geometry (geometry on the surface of a sphere, for instance) and so there is context for a conversation on understanding fundamental assumptions. I am not advocating we introduce examples of non-commutative algebras in the school curriculum. But I do personally believe that being frank and upfront about the role of models is important and pedagogically helpful.

What about 2^0? Well, with no folds, the strip is only one layer thick!

$$0 \text{ folds} \leftrightarrow 1 \text{ layer.}$$

So according to this model, $2^0 = 1$.

What about 2^{-1}? What does it mean to fold a strip of paper in half negative one times? What's an "unfold"?

Well, unfolding would correspond to peeling the paper apart (as though the paper were already folded) to make a strip twice as long but only half a layer thick. So

$$2^{-1} = \frac{1}{2}.$$

Wow!

And 2^{-2} would correspond to peeling the paper apart two times, making a strip a quarter of a layer thick.

$$2^{-2} = \frac{1}{4}.$$

In fact, we can now argue that $2^{-n} = \dfrac{1}{2^n}$ for each positive whole number n.

What a brilliant model!

(I love demonstrating this model with actual strips of paper in front of students and teachers. It invariably excites.)

But my point is that **all models have limitations and will eventually break down!**

Here's a breaking point of this model: a natural question.

What about something like $2^{\frac{1}{2}}$ *?*

Ouch! I have no idea what it means to fold a strip of paper in half half a time. (I can't even conceive of anything to do with paper to get something $\sqrt{2}$ layers thick.)

And this is actually why I think this model is pedagogically brilliant: it propels one and all into a state of excitement over the enlightenment it brings ("Wow! I can see the meaning of 2^{-1}") and then crashes, bringing home so starkly the realization that models have limits and can't solve all our math questions. Everyone is left hanging: *What are we to do?*

Answer: We can now start mathematics.

Step away from the model and ask: *Okay. What feels so natural and "basic" to how exponents work – at least for positive whole numbers? What property of exponents should we choose to believe as universal, as holding for all types of numbers being used as exponents?*

Most people agree with the basic exponent property $2^n \times 2^m = 2^{n+m}$ feels fundamental. It is patently true for positive counting numbers. (For instance $2^2 \times 2^3 = (2\times 2)\times(2\times 2\times 2) = 2^5$.) And at some gut level it feels right to take $2^n \times 2^m = 2^{n+m}$ as a fundamental belief that should hold for <u>all</u> numbers.

Let's go with it!

What then are the logical consequences of this belief?

To make sense of 2^0 let's try substituting $n = 0$ and, say, $m = 3$, into the assumed rule. It gives $2^0 \times 2^3 = 2^3$, that is, $2^0 \times 8 = 8$. We see that $2^0 = 1$ is a forced consequence of our belief.

What does the our fundamental belief say about 2^{-1}?

Well, try $n=-1$ and $m=4$, say. Then we have $2^{-1}\times 2^{4}=2^{3}$, that is, $2^{-1}\times 16=8$, forcing us to set $2^{-1}=\frac{1}{2}$.

Question: How would you demonstrate that 2^{-2} must be $\frac{1}{4}$ as a consequence of this belief? How would you demonstrate that 2^{1} must be 2?

We can now make sense of the troublesome $2^{\frac{1}{2}}$ too. Choosing $n=\frac{1}{2}$ and $m=\frac{1}{2}$ gives $2^{\frac{1}{2}}\times 2^{\frac{1}{2}}=2^{1}$. Since $2^{1}=2$, we have that $2^{\frac{1}{2}}$ is some number that multiplies by itself to give 2. It must be that $2^{\frac{1}{2}}=\sqrt{2}$.

Question: How would you establish that $2^{\frac{1}{3}}$ must be $\sqrt[3]{2}$?

Question: How might you establish that $\sqrt[5]{2^4}$ and $\left(\sqrt[5]{2}\right)^4$ both deserve to be called $2^{\frac{4}{5}}$?

Question: How would you make sense of $2^{1.4}$? Of $2^{1.41}$? Of $2^{1.414}$? Of $2^{\sqrt{2}}$?

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Arithmetic models are certainly powerful tools for tapping into our intuition and beliefs about the properties of numbers. But in the end, we must step back from the models and decide which properties a model might highlight that seem relevant to a universal play of numbers. Which properties of arithmetic, in the end, should we just choose to believe, and what are the logical consequences of those beliefs?

The mechanics of arithmetic and algebra is, by-and-large, perceived as edict and law in the K-12 experience. But by being upfront about the nature of models with our (upper grade) students, we can engage a human mindset towards mathematics. We can foster perspective, flexibility, and powerful meta-cognition.

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