

Curriculum Inspirations

Inspiring students with rich content from the
MAA American Mathematics Competitions



Problem Solving Strategy Essay # 2:

Do Something

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Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

This is the second essay in a series to give credence to this claim on the MAA website, www.maa.org/math-competitions. For over six decades, AMC has been creating and sharing marvelous stand-alone mathematical tidbits. Take them out of their competition coverings and see opportunity after opportunity to engage in great conversation with your students. Everyone can revel in the true creative mathematical experience!

I personally believe that the ultimate goal of the mathematics curriculum is to teach self-reliant thinking, critical questioning and the confidence to synthesize ideas and to re-evaluate them. Content, of course, is itself important, but content linked to thinking is the key. Our complex society is demanding of the next generation not only mastery of quantitative skills, but also the confidence to ask new questions, explore, wonder, flail, innovate and succeed. Welcome to these essays!

Let's make concrete and explicit the benefit obtained from reflecting on, in a deep and joyous way, the mathematical richness a single AMC problem can offer. There are explicit curriculum connections. There are explicit opportunities for original thinking and intellectual play. There are opportunities to ask new questions and to explore new mathematical landscapes. There is human joy in experiencing the true organic and creative nature of mathematics. Do you ever hear those adjectives to describe math? (**ORGANIC MATH? COMPETITON MATH? HUH?**)

It is very easy as educators, especially under the pressure of mandated testing, to unknowingly give the impression to students that all mathematics is geared towards numerical values and answers to "what" questions: *What are the vertex-coordinates of this parabola? What is the measure of angle A? What is the derivative of $\sin^2 x$?* Yet numerical answers are usually secondary to the mathematical story at hand.

It is the *process* of reaching those answers that counts, and this is by far the most interesting aspect. A mathematician—and the human mind itself—naturally wonders about "why" and "what if" questions. (Alas, such questions are hard to grade and tough to write for a multiple-choice test! Hmm.)

At first glance, mathematics competitions tend to reinforce the same mis-view of mathematics. The questions asked often have numerical answers (and so are chiefly "what" questions) and usually come with a choice of numerical values to consider. This makes it clear to the student that she is answering questions that have already been answered by someone else, and that

she is going to be judged as right or wrong. Moreover, there is little or no time for mulling—which is deeply unfortunate as mulling is a fundamental feature in doing true mathematics!

Some students are excited and motivated by the challenge of speed and the sense of competition. Some are not. Some competitors will recognize and value the joy of the “process” in answering a competition problem. Some might not.

This letter is not written as a competition training piece. **This, like all the letters, is an invitation to find the joy in the “whys” and “what ifs” an interesting mathematical idea can offer. We model how to share that joy with all students.**

The AMC has a vast pool of interesting mathematical questions each full of deep and exciting ideas. We show how we can use them as a springboard for mulling, intellectual exploration, and joyful and mindful mathematics.

Let go of any need for speed.

Revel in the experience of exploration, of feeling flummoxed but pushing on nonetheless.

Work to create a safe culture for your students, one in which they can share and/or explore wild ideas that might only be false leads. Follow the leads.

Don't just settle for the “what” answer. Celebrate and mull on tangential ideas, new questions, new connections, new thoughts.

There really is no hurry. True learning comes from having a sense of ownership of one's mathematics – and this takes its own human time.

OUR CHALLENGE TODAY

We contemplate **query 18** from the **2011 MAA AMC 12a** competition.

Suppose that $|x+y|+|x-y|=2$. What is the maximum value of x^2-6x+y^2 ?

I am also unsettled by the idea of maximizing a formula with two variables.

Don't discount the psychological benefit of articulating your worries. Their potency is diminished in doing so.

After a breath we might now be ready for ...

How one might go about thinking about this problem.

STEP ONE: Be honest about your humanness and acknowledge your emotional reaction in first reading the problem.

STEP TWO: Do something

I'll just say it: My first look at this challenge came with an exclamation of panic!

List the features of the problem that induce an emotional reaction.

I personally find the equation with absolute values scary. I really don't have any immediate sense of what that equation is doing/describing/saying.

This is where it is important to let go of the sense of a need for speed. This is NOT a competition. This is NOT a mandated test question. This is a human mathematical experience that is going to take as long as it takes! There will be false starts, a sense of flailing, and a feeling of being in the dark. Honor that process. Running this exploration as a class discussion can dismiss the feeling of any need for speed and “being right” right off the bat.

It is clear that there are two parts to this problem that require attention: making sense of $|x + y| + |x - y| = 2$ and doing something of meaning with $x^2 - 6x + y^2$.

Let's tackle the first equation first. To "do something" means to ask ...

What is the question about?

What do I know about this topic?

Can I "do" what I know to do in this situation?

The equation $|x + y| + |x - y| = 2$ is about absolute values. (State the obvious. It helps!) And I know that absolute values "make quantities positive." (Well, I should say "non-negative" just in case I am dealing with the number zero – but I don't want to be pedantic right now. I need to get a line of thought sorted out first.)

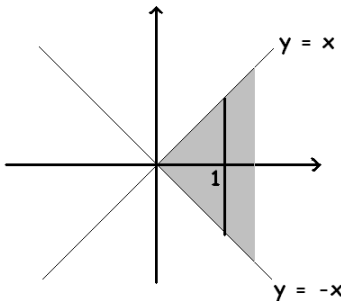
We have two quantities $x + y$ and $x - y$ that are each being made positive. And this makes me think of different cases to examine:

- i) $x + y$ and $x - y$ are each positive;
- ii) they are each negative; and
- iii) one is positive, the other is negative. (Or should I say four cases – split this third case into two?)

I really don't know where this is taking me, but I am just following my nose while keeping my fingers crossed.

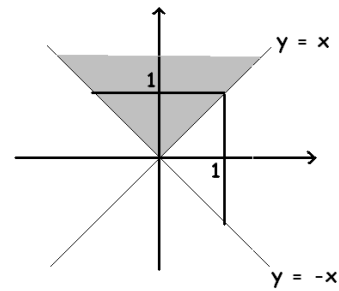
The first case: *What does it mean for $x + y$ and $x - y$ to each be positive?*

Well ... we have $x + y \geq 0$ and $x - y \geq 0$. That is, we have $y \geq -x$ and $y \leq x$. This is the region in the plane bounded by two diagonal lines through the origin.

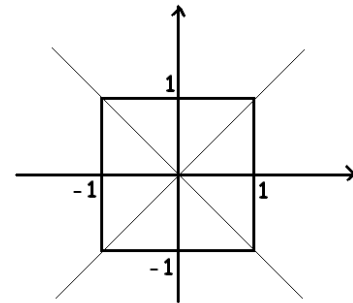


In this region $|x + y| + |x - y| = 2$ reads $x + y + x - y = 2$, that is, $x = 1$. This is a vertical line segment within that region.

A second case: This time let's consider $x + y$ positive and $x - y$ negative. We have $y \geq -x$ and $y \geq x$, and $|x + y| + |x - y| = 2$ reads $x + y - (x - y) = 2$, that is, $y = 1$. We have a horizontal line segment in the region of the plane in the upper section bounded by the diagonal lines.



It won't be a surprise now, and one can check, that the other two cases also yield horizontal and vertical line segments and we see that $|x + y| + |x - y| = 2$ describes a square! Wow!



This feels like an accomplishment, but I am not sure what it does for us.

When stuck, try rereading the question.

I just did this and I am reminded that there is a second piece to attend to: *maximize* $x^2 - 6x + y^2$. What does that mean?

The formula $x^2 - 6x + y^2$ gives different values for different input points (x, y) . We want to find the largest value that can come out when (x, y) satisfies $|x + y| + |x - y| = 2$, that is, when (x, y) is a point on the square. Does that help? Not really, but I did reread the question.

When stuck, just do something!

What can I do with $x^2 - 6x + y^2$? What do I know about things like this?

This equation reminds me of quadratics from algebra II. Maybe I can complete the square:

$$\begin{aligned}x^2 - 6x + y^2 &= x^2 - 6x + 9 + y^2 - 9 \\ &= (x-3)^2 + y^2 - 9\end{aligned}$$

Okay. But I don't know what to do next.

When stuck, try rereading the question.

“Maximize $x^2 - 6x + y^2$.” Well, that is the same as maximizing $(x-3)^2 + y^2 - 9$. That is, we need to make this quantity as large as possible.

Well, I might as well make $(x-3)^2 + y^2$ as large as possible and then subtract 9 from its largest output.

Okay ... What next?

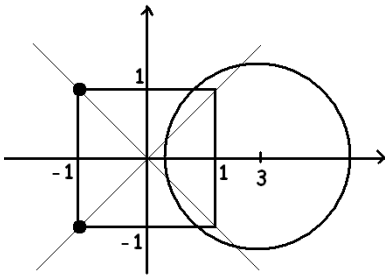
When stuck, just do something!

Aaagh! Okay .. What does $(x-3)^2 + y^2$ remind me of?

The equation of a circle – except it is missing the radius term. (I really do feel like I am struggling for ideas.)

$$(x-3)^2 + y^2 = r^2$$

This is the equation of a circle centered at $(3,0)$ with radius r .

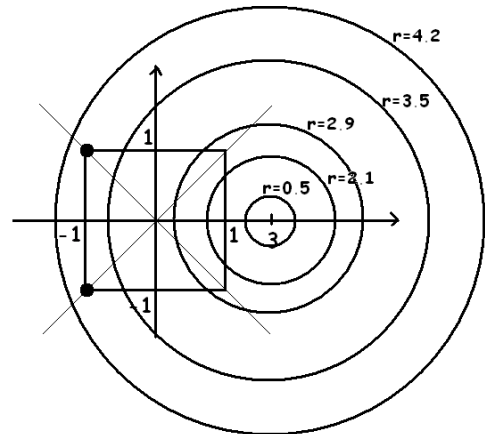


When stuck, try rereading the question.

We want to maximize $(x-3)^2 + y^2$. (That's my current version of the question.)

By Jove ... It's a geometry problem! $(x-3)^2 + y^2$ is just my r^2 , and we want to maximize r^2 among all points on the square. That is, we want a point on the square that

corresponds to a point on the circle, center $(3,0)$, of maximal radius.



That is, we want the point on the square furthest away from $(3,0)$. We see that the points $(-1,1)$ and $(-1,-1)$ are both the maximal distance from $(3,0)$, and by Pythagoras's theorem their distance is $r = \sqrt{4^2 + 1^2} = \sqrt{17}$. Thus $(x-3)^2 + y^2 = r^2$ has largest value 17 among all the points on the square, and $x^2 - 6x + y^2$ has largest possible value $17 - 9 = 8$. Crazy!

COMMON CORE STANDARDS and PRACTICES:

Now that we have solved this AMC problem we can see that it is closely connected with a slew of CCSS-M standards and curriculum skills:

A-SSE-2: See structure in expressions

A-SSE-3: Write equations in equivalent forms to solve problems

A-REI-10,11: Represent and solve equations graphically

G-GPE-1: Express geometric properties with equation

We reviewed **absolute values** and the **equation of a circle**, and we **completed a square**.

We are also right on the mark with the following practice standards:

MP1: Make sense of problems and persevere in solving them.

MP2: Reason abstractly and quantitatively.

MP3: Construct viable arguments and critique the reasoning of others.

MP7: Look for and make use of structure.

and maybe even...

MP5: Use appropriate tools strategically.

[Have students plot $(x-3)^2 + y^2 = r^2$ for different values of r and create the diagram on the left themselves and deduce its meaning.]

DECONSTRUCTING THE PROBLEM:

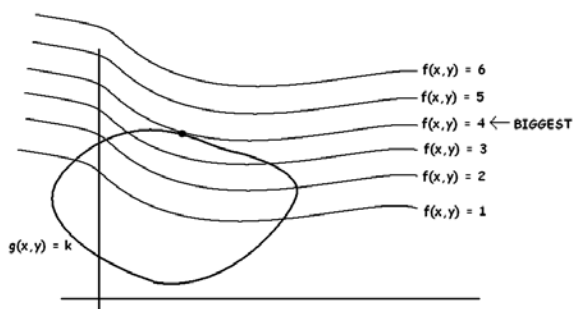
A great sense of accomplishment comes from solving a challenge. Good for us! But true progress and innovation in science and business comes from pushing boundaries, asking new questions, and forging interesting paths of one's own devising. Let's help our next generation of citizens do that too!

How can we use the problem to inspire original enquiry and discovery?

ASK ... What was the big idea behind the problem? (The answer is usually hidden in the part of the process that felt like a surprise.)

Our solution had an unexpected connection to geometry. We worked with two curves in the plane – a square and a circle – and attempted to find the largest value of some parameter that altered the one of the curves – in our case the radius of the circle – and still had the curves overlapping.

That is we worked with two curves, each given by a formula with x s and y s, say $f(x, y) = M$ and $g(x, y) = k$, and we wanted to find the biggest value of M for the points on the curve $g(x, y) = k$.



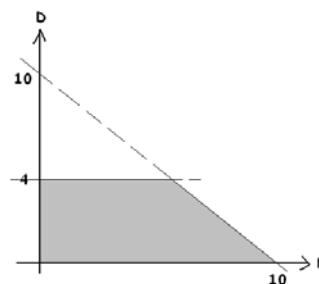
Curriculum Connection:

LINEAR PROGRAMMING

Some algebra II courses have students play with aspects of "linear programming." Here's a typical textbook question:

Dr. T. is a working mathematician, but he bakes desserts in his evenings and sells them to the local café the next day. He makes pavlova and dacquoise. In an evening he can bake a maximum of 10 desserts, but at most four of them can be dacquoise. He makes \$2 profit on each pavlova he bakes and \$3 on each dacquoise. How many of each dessert should he make in an evening so as to maximize his profit?

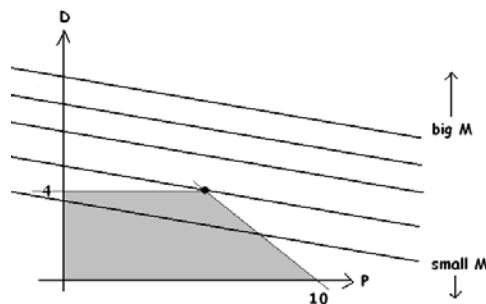
Answer: Let P and D represent the quantity of each dessert. We have the constraints $P \geq 0$, $D \geq 0$, $P + D \leq 10$ and $D \leq 4$. These describe a region in the pavlova/dacquoise plane.



Let M be the profit made. We want to maximize this value $M = 2P + 3D$. Now for each value of M ,

$D = -\frac{2}{3}P + \frac{M}{3}$ is the equation of a line of negative slope.

The placement of this line varies as we change the value of M .



It is now clear that the largest value of M from the region we are considering occurs for $D = 4$ and $P = 6$. The maximum profit is $2 \times 6 + 3 \times 4 = \24 - a nice supplement to his usual income!

QUESTION: Is the focus on “maximum” key? What if Dr. T. wanted to minimize his profit? What is the obvious intuitive answer to the minimum problem re how many desserts he should make? Does the math give the same answer?

Suppose that $|x + y| + |x - y| = 2$. What is the minimum value of $x^2 - 6x + y^2$?

GIVE POWER TO YOUR STUDENTS:

Have students create their own problems based on the idea of maximizing or minimizing one formula with a constraint given by a second formula. They can do it!

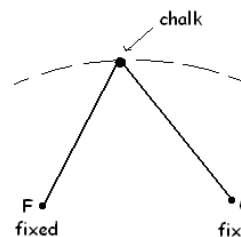
OR

Have them “see through” a whole host of scary looking questions based on the work they know on the shapes of graphs. For example: *What is the largest a for which the equations $y = |x - a|$ and $x^2 + y^2 = 1$ have a common solution?*

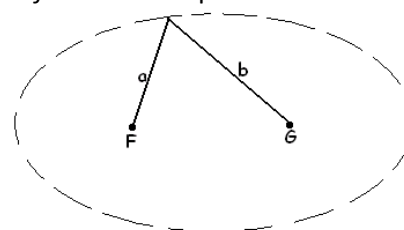
Curriculum Connection:

THE REFLECTION PROPERTY OF AN ELLIPSE

Suppose we fix to the two ends of a string at points F and G on a chalkboard and pull the string taut with a piece of chalk.



As we swing the chalk about the two points, while keeping the string taut, we start to trace a curve. One full rotation produces an oval shape, an *ellipse*. The special points F and G are the *foci* of the ellipse.



$a + b = \text{length of string, always}$

Any room with walls curved in the shape of an ellipse has the astounding property that a whisper uttered at one focus can be heard by anyone located at the second focus: sound waves from one focus bounce off the curved wall and head directly to the second focus. Moreover, the distance each wave travels is the same, so all arrive synchronized!

The Mormon Tabernacle in Salt Lake City and the Whispering Gallery in the United States Capitol building, Washington, D.C., are each ellipses and so have this property.

Amazingly... This reflection property can be proved precisely by the method of this letter! It can be seen as a minimization problem for two curves in the plane. Watch the short video www.jamestanton.com/?p=1189 for all the details!

A SECOND SURPRISE:

Continuing to reflect on our solution I recall now that I was also surprised by the fact that the equation $|x + y| + |x - y| = 2$ describes a square!

FUN: Sketch $\max(|x|, |y|) = 1$.

Here "max" represents the larger of the two given numbers. For example, $\max(4, 7) = 7$ and $\max(4.6, 4.6) = 4.6$.

The fact that two very different equations can describe the same curve reminds me of another curriculum connection:

Curriculum Connection:

THE EQUATION OF A CIRCLE

We teach our students that the equation of a circle with center (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$. (This is just Pythagoras's theorem at work.)

CHALLENGE:

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Show that: $(x - a_1)(x - b_1) + (y - a_2)(y - b_2) = 0$ is the equation of the circle with \overline{AB} as diameter!

Afterthought ... If this is true, then $(x - 2)(x - 5) + (y - 4)(y - 9) = 0$ is the equation of the circle with diameter \overline{AB} where $A = (2, 4)$ and $B = (5, 9)$. Or is that with $A = (2, 9)$ and $B = (5, 4)$? Oooh ... Does this mean something wrong?

What curriculum connections do you notice with this AMC problem and its solution?

When you discuss this problem and its solution with your students, what connections do they notice? What new ideas do they suggest for further exploration?

Mathematics is an ongoing discussion. Textbook questions, classroom discussions, and competition problems are not closed, finite experiences. They are invitations for conversation, exploration and further discovery. Let's keep open that door of conversation for our students.

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.