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The Geometry of Adding Up Votes

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The 2000 U.S. presidential election was a source of interesting politics—and mathematics. George W. Bush was elected president despite losing the popular vote to Al Gore (47.87 percent to 48.38 percent) because he won a majority of the electoral votes (271 to 267). The outcome was an example of an *aggregation paradox*.

As we will show, the 2016 Arkansas and Alabama Democratic primaries are more recent examples of aggregation paradoxes. However, these paradoxes didn't occur when a winner-take-all method was used (as when a state's electoral votes are awarded to the popular vote winner), but when proportional methods were used to allocate delegates. Moreover, we will show the geometry behind the paradoxes.

Apportionment in the Democratic Primary

The 2016 Democratic and Republican primaries were two of the most contentious and unusual primaries in recent history.

In advance of the 2016 primaries, many states' Republican parties changed their rules for how delegates were awarded, from a winner-take-all method to some type of proportional method. This was not to rig the system, as Donald Trump suggested, but perhaps to extend the primary season to better vet the candidates and keep them in the news. However, there was little commonality between how each state's Republican party decided to convert votes to delegates.

In contrast, the Democratic Party uses the Democratic Delegate Selection Rules (DDSR) for each of its state primaries (Section 13, Part D, *democrats.org*). We focus on the Democratic primary because there are consistent rules and there were two main candidates. Although the two-candidate situation is easier to visualize geometrically, the same ideas can be used to explain paradoxical behavior for any number of candidates.

Let's look at how the DDSR were applied in the 2016 Arkansas Democratic primary. Arkansas is divided into four congressional districts, and the delegates for each district are awarded based on the votes in the district. In District 1, 50,231 people voted. We first calculate the percentage of the vote each candidate received, and we eliminate the candidates who received less than 15 percent (and their votes). This leaves Hillary Clinton, Bernie Sanders, and their 47,513 combined votes (called *qualified votes*). See table 1.

District Number	Qualified Vote	Total Delegates	Clinton Vote	Clinton Quota	Clinton Delegates	Sanders Vote	Sanders Quota	Sanders Delegates
1	47,513	5	34,358	3.616	4	13,155	1.384	1
2	66,261	6	48,495	4.391	4	17,766	1.609	2
3	38,375	4	21,087	2.198	2	17,288	1.802	2
4	54,607	6	38,858	4.27	4	15,749	1.73	2

Table 1. The 2016 Arkansas Democratic primary results by district.

The *quota* for each candidate is determined by multiplying the percentage of qualified votes for the candidate by the number of delegates to be awarded. Clinton's quota is $\frac{34{,}358}{47{,}513} \cdot 5 \approx 3.616$, while Sanders's is 1.384. Next, we round down each quota. So, Clinton and Sanders initially receive three delegates and one delegate, respectively.

At this point the DDSR say to distribute the remaining delegates to the candidates in order of their fractional remainders. This method is known as *Hamilton's method* (named for Alexander Hamilton) and was the subject of the first presidential veto, by George Washington. Because District 1 has five delegates, there is one still unawarded, and Clinton receives the delegate because $0.616 > 0.384$. In fact, when there are only two candidates, we compute the number of delegates by rounding the quotas in the usual fashion.

Arkansas had 21 such delegates to award. The result for each district appears in table 1. What would have happened if the 21 delegates were awarded using the DDSR but statewide, not district by district?

Using the 206,756 qualified votes, Clinton's quota would be 14.504 and Sanders's quota would be 6.496. Clinton would receive 15 delegates because $0.504 > 0.496$. However, 15 does not match the 14 $(= 4 + 4 + 2 + 4)$ delegates she received from the four districts. In Arkansas, as with Bush-Gore, we have the counterintuitive—or paradoxical—outcome that the sum of the parts is not equal to the whole.

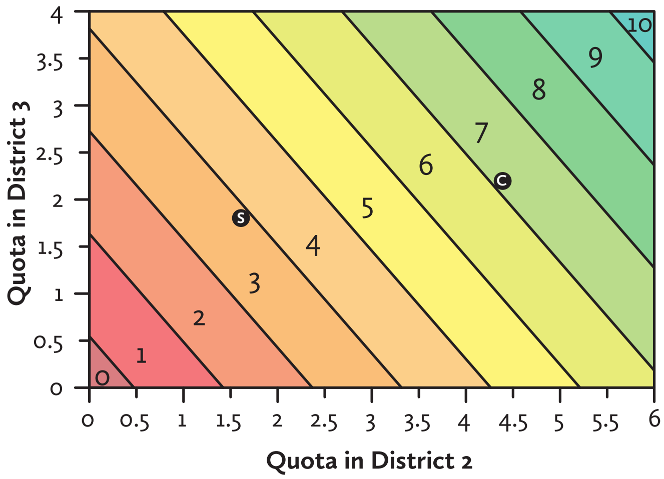


Figure 2. Clinton receives seven delegates and Sanders receives three if we allocate the delegates using the merged districts.

The Underlying Geometry

It is tough to visualize the aggregation paradox from the Arkansas primary geometrically because there are four districts. For simplicity, let's use only Districts 2 and 3, which together exhibit the same paradox.

Clinton received four delegates from District 2 and two from District 3 for a total of six of the 10 delegates. The horizontal and vertical strips in figure 1 show the regions in which the quotas round to specific integers for Districts 2 and 3. They also show Clinton's and Sanders's quotas. Moreover, we can treat the quotas from the two districts as x- and y-axes, producing a 6×4 rectangle. Then the candidates' quotas are ordered pairs, and the rectangle shows the sum of the delegates from the two districts.

If we merge the districts, there are 104,636 qualified votes. Clinton's 69,582 votes yield a quota of $\frac{69{,}582}{104{,}636} \cdot 10 \approx 6.650$; Sanders's quota is 3.350. Clinton's quota would be rounded up, giving her seven delegates.

Allocating delegates based on the total population can be visualized as a different partition of the 6×4 rectangle. Fixing a

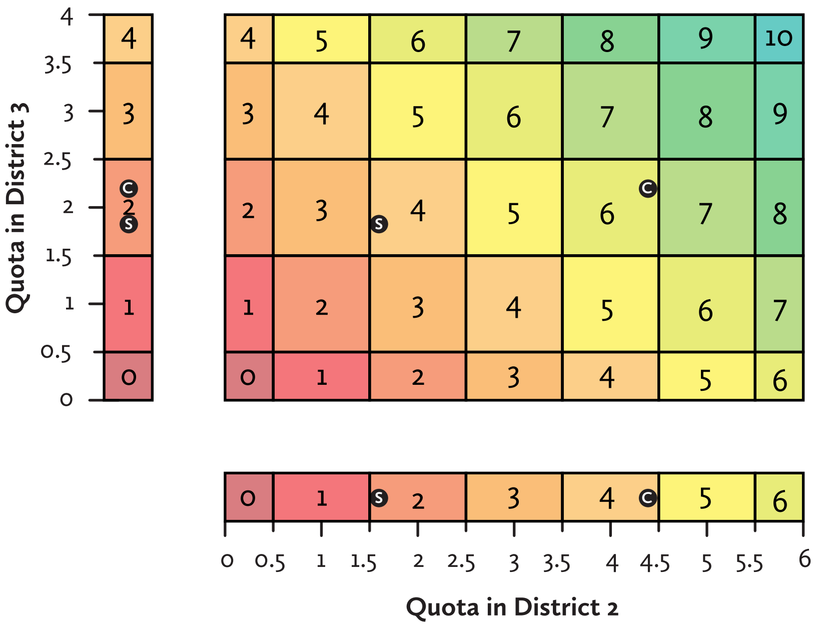
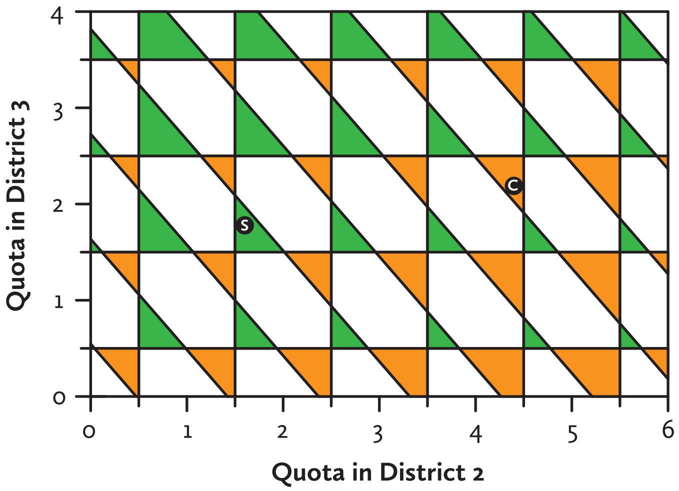


Figure 1. Clinton receives six delegates and Sanders receives four if we allocate the delegates separately in each district.



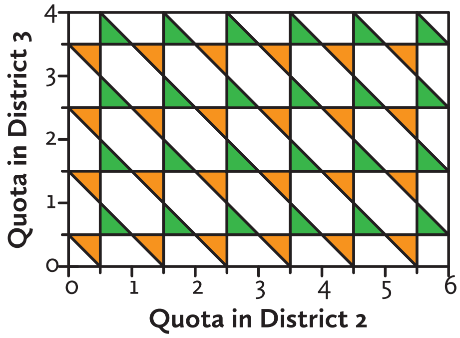


Figure 3, left. Visualizing the aggregation paradox.

Figure 4, above. In a perfect world, the probability of the aggregation paradox is 0.25.

candidate, for district i, let q_i be the candidate's quota, p_i be the number of votes cast for this candidate, and Q_i be the number of qualified votes. The candidate's quota for the merged districts is

$$\frac{p_2 + p_3}{Q_2 + Q_3} \cdot 10.$$

This quota rounds to n when

$$n - 0.5 \leq \frac{p_2 + p_3}{Q_2 + Q_3} \cdot 10 < n + 0.5.$$

To partition the 6×4 rectangle, rewrite the inequality in terms of q_2 and q_3 by using their definitions:

$$q_2 = \frac{p_2}{Q_2} \cdot 6$$

and

$$q_3 = \frac{p_3}{Q_3} \cdot 4.$$

The inequalities create the diagonal bands in figure 2.

In figure 3, we overlay the partitions from figures 1 and 2 to see when the two methods give different numbers of delegates and yield an aggregation paradox. These regions are colored green and orange—green indicates when the merged districts result in one fewer delegate, and orange indicates when it results in one more delegate. As predicted by our calculations, Clinton's quota falls in an orange region and Sanders's quota is in a green region.

To give an idea of the likelihood of an aggregation paradox, we compute the fraction of the 6×4 region that is orange or green. (Notice that the areas are equal because the figure exhibits odd symmetry through (3,2).) We conclude that if each ordered pair of quotas is equally likely, then the probability of an aggregation paradox is about 0.2734.

For elections with $d > 2$ districts, the geometry is similar, except the rectangular region in figures 1 to 3 becomes a d-dimensional box. However, for $c > 2$ candidates, the geometry is much more complicated because the rounding depends on the distribution of quotas and there may be more than one extra delegate to distribute.

In a Perfect World

In a perfect world, each voter's vote is worth the same fraction of a delegate. For this to happen, the number of delegates awarded in a district would be proportional to the total number of qualified votes cast in the district. In the Arkansas primary, Districts 2 and 3 are represented by six and four delegates, respectively, so District 2 *should* have $6 / 4$ the number of qualified votes cast in District 3.

In this case, one-quarter of the 6×4 rectangle is green or orange, as in figure 4. This means that the likelihood of the aggregation paradox, if a uniform distribution were used, is 0.25. This special case generalizes whenever the ratio of the qualified votes cast for the two districts is in the same proportion as the ratio of the delegates for the districts.

Proposition. For any pair of districts, 1 and 2, the likelihood of the aggregation paradox is 0.25 when the ratio of the number of qualified votes Q_1 / Q_2 is equal to the ratio of the number of delegates d_1 / d_2.

Twin Voters

The aggregation paradox may not be too surprising because the voters in each district may behave very differently. What if there were two identical districts—

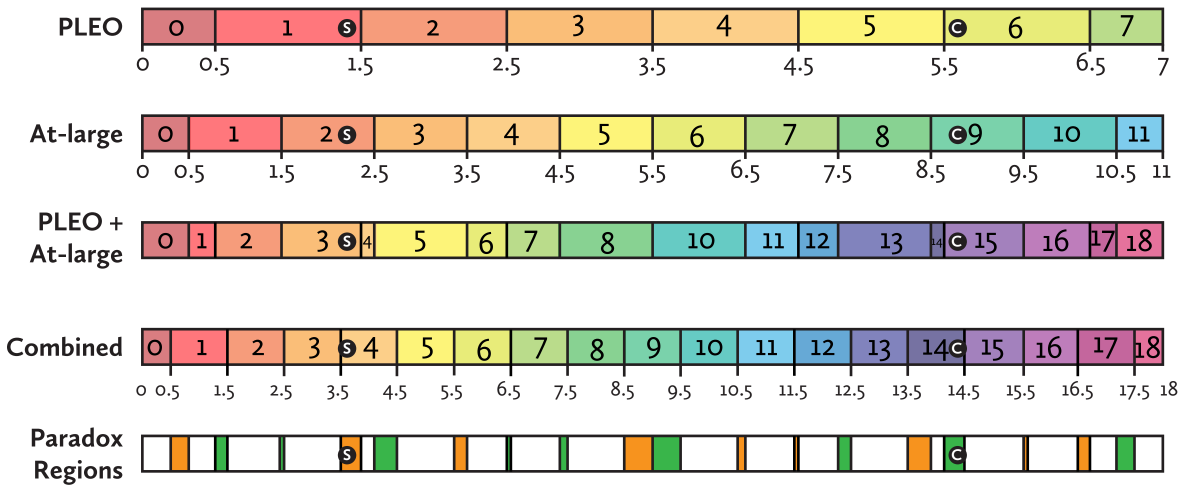


Figure 5: Representing the likelihood of an aggregation paradox in the Alabama Democratic primary.

population and numbers of delegates—and each voter in one district had a twin voter in the other district who voted exactly the same way? The outcome in each district would be identical, including the number of delegates earned by each candidate.

Even in this Xerox-machined world, the aggregation paradox can still occur! For example, suppose that 0.56 of the qualified voters vote for candidate A in each district, and each district has 10 delegates. Then, candidate A would receive six delegates from each district (by rounding 5.6 up to 6). If the districts were merged and the 20 delegates were awarded based on 0.56 of the qualified votes, then A would have a quota of 11.2, which would round down to 11. Because $12 \neq 11$, the paradox occurs even in this carbon-copy world!

This toy example may sound strange, but a similar situation occurred in Alabama's 2016 Democratic primary, except that the district sizes were different (and Alabama does not have two districts with twin voters!). The Democratic primary awards delegates not only at the district level, but also at the state level. The two types of statewide delegates, PLEO (party leader and elected official) and at-large delegates, are awarded based on qualified votes from the entire state. However, the PLEO and at-large delegates are awarded separately. An aggregation paradox can occur if the sum of the delegates is different than if they were awarded together as one block. This happened in Alabama.

There were 398,144 votes cast in the Alabama 2016 Democratic primary, but only 386,335 were qualified votes. Clinton received 309,932 votes, and Sanders received 76,403. There were seven PLEO delegates for which Clinton's and Sanders's quotas were 5.616 and 1.384, respectively; this resulted in Clinton earning six delegates. There were 11 at-large delegates for which Clinton's and Sanders's quotas were 8.825 and 2.175, respectively; so Clinton received nine delegates. Hence, Clinton received 15 of the possible 18 statewide delegates.

If the 18 delegates were awarded together, then Clinton's quota would be 14.440, while Sanders's quota would be 3.560. Clinton would receive only 14 of the statewide delegates. Because $14 \neq 15$, this is another example of the aggregation paradox.

The geometry is a little different. In the first four segments in figure 5, we see what happens when the PLEO and at-large quotas are rounded separately and when their combined quota is rounded. To see which quotas yield paradoxes, we could draw a figure like that in figure 4. But because the statewide districts use the same vote totals to determine the quotas, the quotas would fall on a diagonal line in the rectangle.

Clinton's quotas satisfy

$$Q_{\text{PLEO}} = \frac{309,932}{386,335} \cdot 7$$

and

$$Q_{\text{at-large}} = \frac{309,932}{386,335} \cdot 11,$$

hence

$$Q_{\text{PLEO}} = \frac{7}{11} \cdot Q_{\text{at-large}}.$$

So, the line for Clinton's quotas goes through the origin and (11,7), and likewise for Sanders. The bottom segment in figure 5 shows where this line intersects the green and orange regions. The green regions indicate that the delegates awarded together would be one less than when they are awarded separately, and the orange regions represent when they are one more.

If a point is selected at random, the probability of being in a green region, or by symmetry an orange region, is 0.1266. Thus, the probability of a paradox is just over a quarter.

Super delegates complete the roster of delegates in the Democratic primary. These are designated party leaders who cast a vote for the candidate of their choosing.

The Presidential Election

The general election in November is akin to having 51 districts—the states and the District of Columbia. Just as in Arkansas, and like the Bush-Gore election, we know that the aggregation paradox can occur. All but two states award their electoral votes on a winner-take-all basis. Nebraska and Maine allocate one electoral vote to the popular vote winner in each congressional district and two electoral votes to the popular vote winner of the state.

The Michigan legislature debated awarding electoral votes by district and proportionally by the statewide vote (see J. Oosting, "Michigan Panel Debates Changes to Presidential Election System, Electoral College Votes," *MLive.com*, September 24, 2015, *http://bit.ly/21svv3z*). Proponents for this initiative, and for one in Pennsylvania, believe that allocating electoral votes by a winner-take-all method marginalizes voters who vote for less popular candidates.

In contrast, there has also been support for the National Popular Vote bill in the Michigan legislature (see *nationalpopularvote.com*). So far, 11 states with 165 electoral votes have passed this bill into law, committing a state's electoral votes to the nation's popular vote winner. The states' laws are enacted only when states with a majority of the 538 electoral votes pass the bill into law. If the National Popular Vote bill is passed by enough states, then the popular vote winner would by law become the electoral vote winner, eliminating outcomes such as that from the Bush-Gore election. One thing is for sure, there are always more elections, and the elections are a good source for interesting mathematics.

Further Reading

We, with K. Geist, analyzed the Democratic Delegate Selection Rules in 2008 (Apportionment in the Democratic primary process, *Math. Teacher* **104** no. 3 [2010] 214–220). The Green Papers (*thegreenpapers.com*) is a good source of data. Bradberry considered geometric approaches to other apportionment paradoxes (A geometric view of some apportionment paradoxes, *Math. Magazine* **65** [1992] 3–17). Saari looks at the geometry of elections more generally (*Chaotic Elections*, American Math. Society, Providence, RI, 2001). ■

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Trevor Evans Award

We are happy to announce these recent recipients of the MAA's Trevor Evans Award, which goes to authors of exceptional articles published in *Math Horizons*.

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