

## James Propp

A$s$ the showrunner of my nine-year-old son's paintball birthday party, I expected to face lots of problems. I just didn't expect any of them to be math problems. There were eight boys at the party, including my son and his close buddy, and while those two would've been happy to be on the same team in every four-on-four game, my wife wisely suggested that I set things up so that each boy would be on my son's team the same number of times. In fact, it would be ideal if, over the course of the party, each boy could be each other boy's teammate the same number of times. Then no one would have cause to call "No fair!" the way nine-year-olds do.

## Randomness

One approach to fairness is randomness. For each game I could divide the boys randomly into two teams of four. In each game my son and his buddy would have a 3 -out-of- 7 chance of being on the same team. So if the boys played seven games, which was just about the number of games they could play in their three-hour window (allowing time for pizza and birthday cake), the expected number of times each pair of boys would be on the same team would be three.

The problem with the random approach is that even though it's good on average, it's sometimes bad. There'd be about a 2 percent chance that my son and his buddy would never be on the same team. And


Figure 1. We represent the eight boys by the eight vertices of a cube.
there'd be an even larger chance that some pair would never be on the same team. If this happened, there'd be a good chance that by the end of the party the kids would have been calling "No fair!"
Thus, I rejected the random approach and wondered if I could design a seven-game schedule for eight boys so that any two boys play on the same team exactly three times. The reader may wish to try this problem before reading further.

## Geometry, Geometry, Geometry!

Because I'm a mathematician, the fact that there were eight, or $2^{3}$, boys at the party screamed "Use geometry!" at me: It seemed natural to assign each of the eight boys to one corner of a cube, as in figure 1, and proceed using geometrical ideas.

The new, geometrified problem is to divide the vertices of a cube into two sets of four in seven differ-

(a)

(b)

(c)

Figure 2. Three ways to divide the eight vertices into two sets of four.
ent ways, so every pair of vertices is in the same set exactly three times (from now on, "three times" means exactly three times). It's helpful to think of painting the vertices in the sets purple and green, as in figure 2.

We could start by slicing the cube with cuts parallel to faces of the cube. The six faces give three ways to cut, so we get the three colorings in figure 2 . We will represent my son by the origin $A=(0,0,0)$, and we will color a vertex purple if the corresponding child is on the same team as the birthday boy.

In figure 2 a , the point $(x, y, z)$ is purple or green according to whether $x$ is 0 or 1 ; in figure $2 \mathrm{~b},(x, y, z)$ is purple or green according to whether $y$ is 0 or 1 ; and in figure $2 \mathrm{c},(x, y, z)$ is purple or green according to whether $z$ is 0 or 1 .

This is a good start, but how should we continue? We need to find the four other ways to paint the vertices.

## Planes with Only Four Points

Finite fields are mathematical systems that mimic some aspects of ordinary arithmetic, but with a key difference: They contain only finitely many numbers. The simplest finite field - and the one we need - has two elements, $\operatorname{GF}(2)=\{0,1\}$ (GF stands for Galois field). It satisfies the properties shown in tables 1 and 2 .

The equation $1+1=0$ may seem strange, but the arithmetic of GF (2) makes sense if you think of 0 and 1 as meaning even and odd, respectively. Then the equation just means that the sum of two odd integers is even.

Part of the power of finitefield arithmetic is that it gives rise to its own geometry. Recall that points in ordinary threedimensional Euclidean geometry can be represented by triples

(a)

(b)

(c)

(d)

Figure 3. Four more ways to divide the eight vertices.

|  | Game | Blue Team | Red Team |
| :---: | :---: | :---: | :---: |
|  | 1 | ABCD | EFGH |
|  | 2 | ABEF | CDGH |
|  | 3 | ACEG | BDFH |
|  | 4 | ABGH | CDEF |
|  | 5 | ADEH | BCFG |
|  | 6 | ACFH | BDEG |
|  | 7 | ADFG | BCEH |

Table 3. The schedule constructed from the colorings in figures 2 and 3.
$x+y+z$ (left to right), with points painted purple or green according to whether the expression equals 0 or 1, respectively.

The colorings in figures 2 and 3 produce the schedule shown in table 3 . If you came up with this same solution, or something like it, without knowing about $\mathrm{GF}(2)^{3}$, take a bow!

## Symmetry, Symmetry, Symmetry!

Our work isn't done yet. How do we know that each pair of boys plays on the same team exactly three times? Or, equivalently, how do we know each pair of vertices of the cube is the same color exactly three times?

We could take the brute force approach and check that this schedule has the desired property, but there are 28 pairs of boys at the party. That would be a fair bit of work!

One way to reduce the number of pairs to check is to use the symmetries of a cube sitting in $\mathbb{R}^{3}$. For instance, let's consider threefold rotational symmetry of the cube about the axis going through A and H . (If you've got a pet cube, Rubik's or otherwise, hold opposite corners between your two index fingers and then give the cube a spin with your thumb.) For any point $(x, y, z)$, a 120-degree rotation about this axis sends $(x, y, z)$ to $(y, z, x)$, a second rotation brings it to $(z, x, y)$, and a third rotation brings it back to $(x, y, z)$.

The seventh coloring (the cube in figure 3d) isn't affected by these rotations. The other colorings are permuted by these rotations: A rotation turns each of the six colorings into one of the other six. This symmetry cuts down the number of cases we need to check. If players $\mathrm{B}=(0,0,1)$ and $\mathrm{D}=(0,1,1)$ play on the same team three times, then the same must be true for players $C=(0,1,0)$ and $G=(1,1,0)$, and similar reasoning
shows that the same must be true for players E and F . This style of reasoning using symmetry has its, roots. in common sense, but it takes some getting used to; that's part of what one picks up in an abstract algebra course.

Using only rotation about the line through A and H, we can reduce the number of cases to be checked from 28 to 10 . If we use more symmetries of the cube, we can bring that down to just three. Wearing symmetryspectacles, we see that we just need to check that our condition holds for a pair of points that share an edge of the cube (such as A and B), that share a diagonal of one of the cube's faces (such as A and D), and that are diametrically opposite (such as A and H).

That (approximately) 10 -fold saving of labor is pretty neat, isn't it? But it's not what I actually did. That's because there are some symmetries in $\mathrm{GF}(2)^{3}$ that aren't present in $\mathbb{R}^{3}$. There's a symmetry operation on $\mathrm{GF}(2)^{3}$ that carries the points A and B to A and D , respectively, and there's another that carries A and D to A and H , respectively.

The symmetry operation on $\operatorname{GF}(2)^{3}$ that carries $\mathrm{A}=(0,0,0)$ and $\mathrm{B}=(0,0,1)$ to A and $\mathrm{D}=(0,1,1)$ is the shear mapping that sends the points of the form $(x, y, z)$ to $(x, y+z, z)$. We know that this symmetry permutes the seven colorings because, being a linear transformation, it sends planes to planes, and the 14 planes given by our seven colorings are all the planes in $\operatorname{GF}(2)^{3}$ ! So, if A and B satisfy the exactly-three-games condition, so must A and D.

Likewise, the shear mapping that sends $(x, y, z)$ to $(x+z, y, z)$ provides a way to see that if A and D satisfy the exactly-three-games condition, so must A and H. Together, these transformations show that there's only a single case to check.

## The Final Trick: Checking No Cases at All

That 27-fold saving of labor is even neater, isn't it? But it's still not what I actually did. Once we know that the answer to the question "How many times is a pair of points the same color?" is the same for all pairs, and once we show that the average value of that answer (as we vary the pair) must be three, we can conclude that in every case, the answer must be three, as we wanted to show! Thanks to symmetry and an averaging argument, there are zero cases to check.

Here's one way to see that the average is three. For each of the 28 pairs of boys, make a row on a piece of paper, and every time two boys play on the same
team, put a mark in that row. For instance, suppose that in the first game we, pit $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D against E, F, G, and H. Then we put tickmarks in rows $\mathrm{AB}, \mathrm{AC}, \mathrm{AD}, \mathrm{BC}$, BD , and CD and rows $\mathrm{EF}, \mathrm{EG}, \mathrm{EH}$, FG, FH, and GH-twelve marks in all. Over the seven games, the number of marks in our tally-chart will be $7 \cdot 12=84$. So the average number per row is $84 \div 28=3$.

The party was a success-so much so that my son's buddy had a paintball party for his birthday too. But his father took a different approach to the fairness problem: At the start of each game, he randomly lined the kids up, counted off "One, two, one, two . . ." and assigned them to teams accordingly. My son liked the other dad's solution a lot better than mine. Sometimes plain old randomness is the way to go. Perhaps the voice I should have heeded was Thoreau's: "Simplicity, simplicity, simplicity!"

## Variations of the Problem

I was lucky that there were eight boys at the party. Had there been six boys, 10 boys, 14 boys, or indeed any even number of boys not divisible by 4 (leaving aside the easy case of two boys), then there'd be no way to come up with a schedule with the number of games being one less than the number of boys.

Try proving that it is impossible to schedule nine five-on-five games for 10 boys so each boy is on every other boy's team exactly four times. Our proof is at the end of the article, and an analogous argument holds when the number of boys is greater than 2 and is congruent to $2 \bmod 4$.

What about 12 boys or 16 or 20 ? In the case of 12 boys, there's a very special solution based on the sporadic finite simple group $M_{11}$ (see Dima Pasechnik's response to my question at http://bit.ly/ProppQues); I certainly wouldn't have come up with that schedule during the first few minutes of the paintball party!

With 16 boys, we can adapt my solution using $\mathrm{GF}(2)^{4}$ instead of $\mathrm{GF}(2)^{3}$. It's believed that for a larger number of boys the problem can be solved whenever the number is divisible by four, but this has not been proved. It's equivalent to one of the oldest unsolved problems in the theory of combinatorial designs: the Hadamard matrix conjecture.

> The party was a success-so much so that my son's buddy had a paintball party for his birthday too.

Finally, here's something I don't know: If the number of players is an even number not divisible by four, is there a fair schedule in which the number of games is not the number of players minus 1 , but twice that? For example, if there are 10 boys, is there an 18-game schedule in which the same two boys are teammates eight times?

Proof that there's no schedule for 10 boys: For the sake of contradiction, suppose we can come up with such a schedule. Let A, B, and C be three of the boys. Let $w$ be the number of times $\mathrm{A}, \mathrm{B}$, and C are teammates, let $x$ be the number of times A and B play on the same team against C , let $y$ be the number of times A and C play on the same team against B , and let $z$ be the number of times A plays against both B and C . Then A and B are teammates $w+x=4$ times, A and C are teammates $w+y=4$ times, and B and C are teammates $w+z=4$ times. Adding the three equations, we get $3 w+x+y+z=12$. On the other hand, there are nine games, so $w+x+y+z=9$. Combining these equations we get $2 w=3$, which is a contradiction, since $w$ is a whole number.

## Further Reading

This piece is a shortened version of one of my blog posts. See http://bit.ly/paintballproblem for a more detailed discussion and references.

> James Propp teaches math at UMass Lowell and does research in combinatorics, probability, and dynamical systems. He blogs on the 17 th of each month at mathenchant.wordpress.com. He also tweets under the name @JimPropp.

> Thanks to Elwyn Berlekamp, Veit Elser, Sandi Gubin, Christian Lawson-Perfect, Joe Malkevitch, Henri Picciotto, and James Tanton.

http://dx.doi.org/10.4169/mathhorizons.25.2.18

