A Pi Day of the Century *Every Year*

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In 2015, Pi Day celebrations reached new heights of enthusiasm. The digits of the date—when written in the American style (3/14/15)—matched the first five digits of π. As this won't happen again until 2115, math enthusiasts dubbed this day the *Pi Day of the Century* and celebrated appropriately. Those who were ultra-enthusiastic celebrated the *Pi Instant* on 3/14/15 at precisely 9:26:53.58979..., a moment in time that includes all the digits of π (see [3]).

As we face future Pi Days, which no longer coincide with five digits of π, we may see a dip in enthusiasm. Our purpose here is to bring good news: We *will* be able to celebrate a Pi Day of the Century again! Living until 2115 would admittedly be a stretch, but that is not the plan we have in mind. Our idea is—of course—a mathematical one.

Where Does π Come From?

A circle, as we well know, is the set of points equidistant from a given point—the center of the circle. The ratio of the circumference of a circle to its diameter does not depend on the circle. This ratio is π.

To compute π we must be able to measure distances and lengths of curves in the plane. Traditionally, we defined the distance between two points $A = (x_1, y_1)$ and

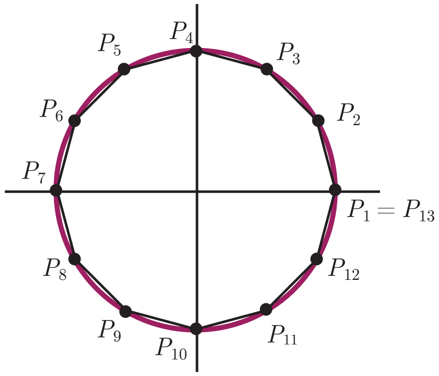

Figure 1. A polygonal approximation of a Euclidean circle.

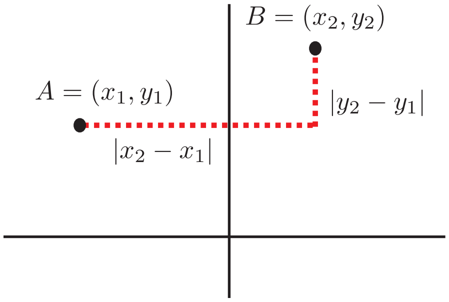

Figure 2. Traveling from point *A* to point *B* using taxicab distance.

$B = (x_2, y_2)$ to be the length of the line segment AB. In other words, we measured distance "as the crow flies." The formula for this *Euclidean distance* is $d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Determining the length of a curve—such as the circumference of a circle—is trickier. To do so, we first approximate the length by picking a finite number of points on the curve, $P_1, P_2, \ldots, P_n$, and summing the distances from one point to the next (see figure 1), $d(P_1, P_2) + d(P_2, P_3) + \cdots + d(P_{n-1}, P_n)$. The arc length of the curve is the supremum over all such values—that is, it is the smallest number that is greater than or equal to the lengths of all these polygonal approximations.

Computing the circumference of a circle in this way is difficult. Through thousands of years of work (which has involved everything from geometric arguments to throwing breadsticks on tiles to harnessing the power of supercomputers), mathematicians have computed trillions of decimal digits of π.

Trading in Crows for Taxicabs

Euclidean distance is a reasonable way to measure distance if you're a crow or if you own a helicopter, but there are infinitely many other ways to measure distance.

One alternative to Euclidean distance is the so-called *taxicab distance*. Imagine we are traveling in a taxicab in a city whose streets form a rectangular grid. To get from point to point, we may drive only along the streets in the grid. By analogy, to measure the taxicab distance from $A = (x_1, y_1)$ to $B = (x_2, y_2)$ in the plane, we add the horizontal and vertical distances between them (see figure 2). So, $d_{tc}(A, B) = |\, x_2 - x_1 \,| + |\, y_2 - y_1 \,|$.

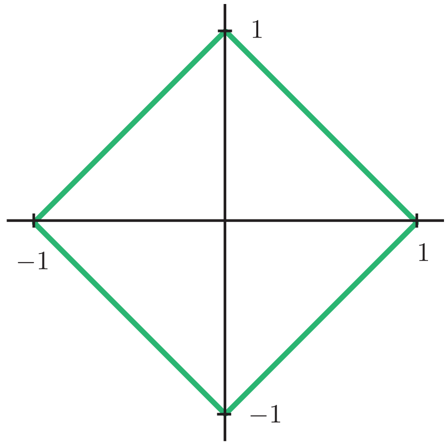

Figure 3. The taxicab unit circle.

Let's adopt the taxicab distance formula and watch the domino effect of changes that ensue. First, circles change shape. For example, the unit circle is the set of points (x,y) that are taxicab distance of 1 from (0,0). Thus, the circle is given by the equation $|\, x \,| + |\, y \,| = 1$ and is diamond shaped (see figure 3).

What is the circumference of this taxicab circle? One is tempted to say that it is $4\sqrt{2}$, but that's if we compute the answer using Euclidean distance.

According to the definition of arc length, we must first take a collection of points on the circle and sum the taxicab distances from one point to the next. Let's focus on the part of the unit circle in the first quadrant. Something wonderfully convenient happens. No matter what points we take, the sum of the taxicab distances is always 2 (see figure 4)! Thus, the arc length of the unit circle in the first quadrant is 2, and the circumference is 8.

So, the ratio of the circumference to the diameter is $8 / 2 = 4$. In fact, for any taxicab circle of radius r,

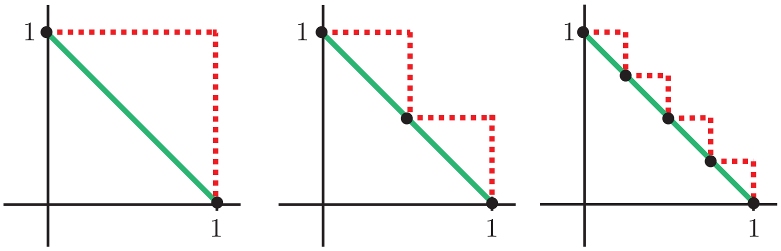

Figure 4. The length of the line segment connecting (1,0) and (0,1), using taxicab distance, is 2.

the ratio of circumference to diameter is $8r / (2r) = 4$. So we write $\pi_{tc} = 4$. Yes, that's correct. Using taxicab distance, the corresponding π-value is simply 4.

The fact that π_{tc} is 4 may be disappointing. Not only is π_{tc} rational, it is an *integer*! There is no mystery, no need for supercomputing power, and no fun in having a π_{tc}-recitation contest. Nevertheless, our interest is piqued. If we can change our distance formula and get circles to be diamond shaped and $\pi_{tc} = 4$, then what other strange things are possible?

Distance (Re)defined

Before we begin cooking up more distance functions, we must introduce some notation and definitions. The most concise way to express a distance function is to define the distance from each point $A = (x, y)$ to the origin (0,0), which we denote $\| A \|$, and which we call the *norm* of A.

We define the distance between the points A and B as $d(A, B) = \| A - B \|$. One can check that the Euclidean distance function arises from the norm $\| (x, y) \|_E = \sqrt{x^2 + y^2}$, and the taxicab distance function arises from the norm $\| (x, y) \|_{tc} = |\, x \,| + |\, y \,|$.

Before moving forward, we mention a useful fact about norms: For any norm $\| \cdot \|$, the arc length of the line segment AB is $\| A - B \|$. We observed this for the taxicab distance, and, with a brief argument, one can prove it for any norm.

Geometry to the Rescue

Finding norms other than the ones we have already mentioned might feel like stumbling around in the dark. Fortunately, there is a visual way to characterize all such norms.

Theorem 1. Norms are in one-to-one correspondence with closed, bounded, convex sets that contain a Euclidean disk around the origin and are symmetric about the origin. Moreover, such a set K is the unit disk for the associated norm $\| \cdot \|_K$.

That's a mouthful! Let's look closely at the parts of this theorem. That K is closed means that it contains its boundary. Bounded means that it does not extend to infinity. Convex means that if A and B are points in K, then the segment AB is in K. To say that K is symmetric about the origin means that if (x,y) is in K, then $(-x,-y)$ is in K.

For instance, theorem 1 rules out the possibility that we could come up with a norm where circles are hearts or stars (a disappointment, to be sure), since these shapes are not convex. Similarly, triangles will never arise as circles under any norm, since they are not symmetric about the origin. However, regular $2n$-gons, ellipses, and a host of other shapes do work.

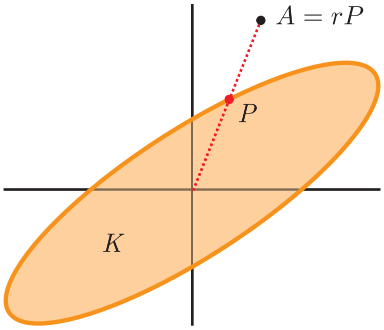

Figure 5. The set *K* determines a norm in which $\| A \|_K = | r |$.

The one-to-one correspondence goes as follows. Given a norm $\| \cdot \|$, the associated set K is the norm's unit disk (the filled-in unit circle). Conversely, suppose K satisfies all the required properties. Then we define a norm $\| \cdot \|_K$ as follows. Every point A can be expressed as $A = rP$, where P is a point on the boundary of K. In that case, define $\| A \|_K = | r |$ (see figure 5).

For example, the regular hexagon H in figure 6 satisfies all the conditions of theorem 1. So it is the unit circle of some norm $\| \cdot \|_H$. We can compute the circumference of this unit circle with respect to $\| \cdot \|_H$. By our previous discussion, the arc length of the line segment with endpoints $A = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $B = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ is the distance between A and B, $\| A - B \|_H = \| (0,1) \|_H = 1$. The second equality holds because (0,1) lies on the unit circle (the boundary of H). Similarly, every side of the hexagon has length 1 with respect to this norm. So, the circumference of the unit circle is 6, and $\pi_H = 3$.

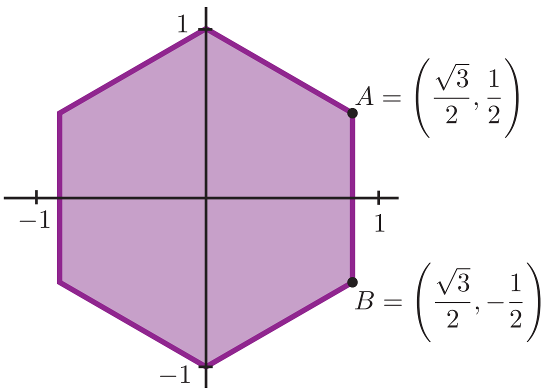

Figure 6. The hexagon *H* determines a norm with $\pi_H = 3$.

Now we have found three different π-values: 3, 4, and the usual $3.14159\ldots$. We could continue with this step-by-step work, but at this rate we will never finish, for, as the following theorem states, there are infinitely many possible π-values!

Theorem 2. Let $\| \cdot \|_K$ be a norm with unit disk K, and let π_K denote the associated π-value.

1) $3 \leq \pi_K \leq 4$.
2) $\pi_K = 3$ if, and only if, K is an affine regular hexagon (the image of a regular hexagon under an affine transformation).
3) $\pi_K = 4$ if, and only if, K is a parallelogram.

Moreover, for any a with $3 \leq a \leq 4$, there exists a set K such that $\pi_K = a$.

In 1932, Stanisław Gołab proved parts (1)–(3) of this theorem (see [2] and also [1] and [4]). The last part of the theorem can be justified succinctly here. Consider the six-sided polygon K_t in figure 7, where $0 \leq t \leq 1$. This set satisfies the requirements of theorem 1, so K_t determines a norm that we denote by $\| \cdot \|_t$. Since the boundary of K_t is polygonal, we can compute the circumference (and hence π_{K_t}) by summing the distances between the vertices. Thus,

$$\begin{aligned}\pi_{K_t} &= \tfrac{1}{2}(2 \| (1,0) - (t,1) \|_t + 2 \| (t,1) - (-1,1) \|_t \\ &\quad + 2 \| (-1,1) - (-1,0) \|_t) \\ &= \| (1-t,-1) \|_t + \| (t+1,0) \|_t + \| (0,1) \|_t \\ &= \| (1-t,-1) \|_t + (t+1) \| (1,0) \|_t + \| (0,1) \|_t \\ &= 1 + (t+1) \cdot 1 + 1 \\ &= 3 + t.\end{aligned}$$

The penultimate equality follows from the fact that $(1-t,-1)$, (1,0), and (0,1) lie on the unit circle, so their norm is 1. Since $0 \leq t \leq 1$, we can realize all values in the interval [3,4] as π-values.

When Can I Eat Pie?

With our new perspective on distances, circles, and circumferences, it is apparent that Pi Day celebrations do not have to be confined to March 14. Generalized π-values begin at 3 and range past 3.31, so we can celebrate Pi Day on any day in March. Moreover, if the reader missed out on the Pi Day of the Century (or the Pi Instant) in 2015, there is no cause for disappointment. For instance, if we want to celebrate a Pi Instant at 12:37:15 on March 25, 2016, we produce a norm

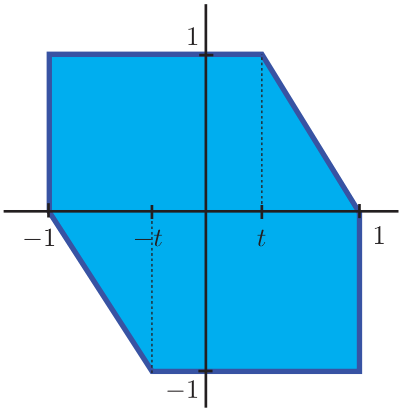

Figure 7. The polygon K_t determines a norm with $\pi_{K_t} = 3 + t$.

$\|\cdot\|_K$ such that $\pi_K = 3.2516123715$ (let $t =$ 0.2516123715 for the set in figure 7), and the party is on!

Finally, many of our friends outside the United States have previously been unable to celebrate Pi Day because their convention for writing dates (with the day first and month second) would place Pi Day on the 31st of April (31/4), a day that does not exist. However, with this new perspective, our international friends can celebrate on the 30th of any month—except February, of course.

In general, changing norms produces not only new π-values, but an entirely new geometry on the plane. This area of study is called *Minkowski geometry*. Can you foresee geometric concepts other than circles that depend on the norm? See [4] for a full discussion of Minkowski geometry both on the plane and in higher dimensions.

Further Reading

[1] J. Duncan, D. Luecking, and C. M. McGregor. On the values of pi for norms on $\mathbb{R}^2$, *College Math Journal* **35** no. 2 (March 2004) 84–92.

[2] S. Gołab, Quelques problémes métriques de la géometrie de Minkowski, *Trav. l'Acad. Mines Cracovie* **6** (1932) 1–79.

[3] J. Rosenthal, Pi Instant, *Math Horizons* **22** no. 3 (February 2015).

[4] A. C. Thompson, *Minkowski Geometry, Encyclopedia of Mathematics and its Applications,* Cambridge Univ. Press, Cambridge, 1996.

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