Building Tangloids

by
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This document provides the mathematical details needed to describe the tangloid curves discussed in the Math Horizons article *All Tangled Up* [1]. That article shows that tangloids are connected sequences of hypotrochoid and epitrochoid pieces. More specifically, imagine that we place a wheel (with pen) inside our tangle and start rolling it counterclockwise along the tangle. Figure 1 motivates the idea that as the wheel traverses a tangle piece the wheel may be "inside" the circle given by the tangle piece, or "outside" the circle. When the wheel is inside the circle, it is rolling counterclockwise. When it is outside the circle, it is rolling clockwise. Each tangloid piece is either a hypotrochoid piece, traced as the wheel goes counterclockwise around the inside of the quarter circle, or an epitrochoid piece, traced as the wheel goes clockwise around the outside of the quarter circle.

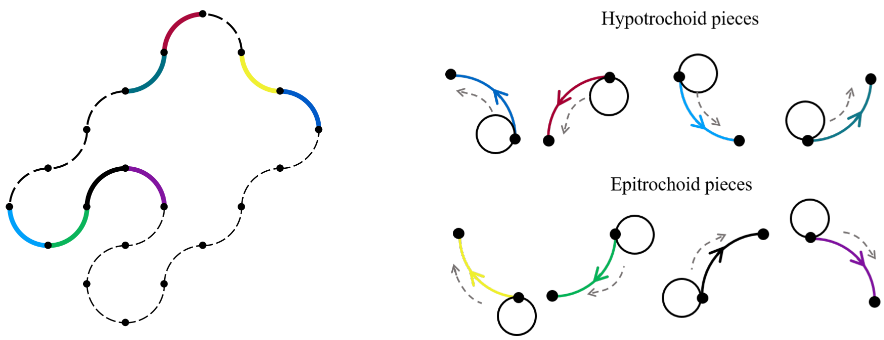

Figure 1: Drawing a Tangloid

By convention, we position our tangle so that one of the tangle pieces has one end at the origin and continues up and counterclockwise. (For the tangle in figure 1, our "first" tangle piece could be the one colored dark blue.) With this "default" position, the wheel always traverses a tangle piece starting at the right, top, left or bottom of the corresponding circle. So for each portion of the tangloid, the wheel traverses a quarter circle of radius R, starting at an initial angular position $\alpha = 0$, $\frac{\pi}{2}$, π, or $\frac{3\pi}{2}$ and continuing counterclockwise around the inside or clockwise around the outside of the circle from that point.

There are well-known parametric equations for hypotrochoids and epitrochoids. Thus, for a given tangle piece, if we can specify:

- the circle corresponding to the tangle piece we are traversing,
- the initial position of the wheel on that circle,
- whether the wheel is "inside" or "outside" circle, and
- where the pen is at the beginning of the roll,

we can immediately write down a parametric description of the corresponding tangloid piece. Therefore, our first task is to give a mathematical description of the tangle curves used to generate tangloids.

Building Tangles: Again, we embed our tangles in the plane by rotating them so that one of the tangle pieces has one end at the origin and continues up and counterclockwise. After that, we assign our first piece the value +1 and proceed counterclockwise around the tangle, assigning each tangle piece a value +1 if it "turns left" and a value -1 if it "turns right." (See figure 2.)

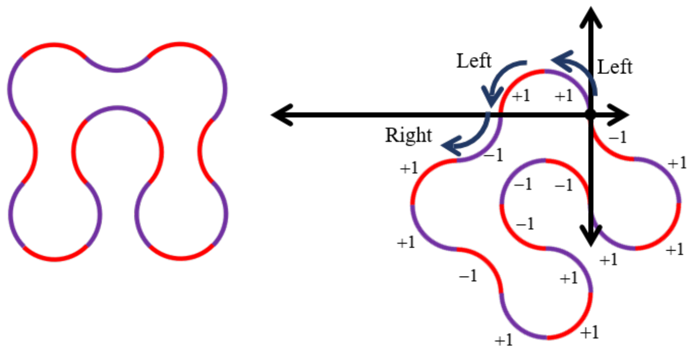

Figure 2: Every tangle can be described with a sequence of plus and minus 1s.

In general, we will use the notation $\langle t_1, t_2, t_3, \ldots, t_n \rangle$ to refer to the vector of ± 1's that describe a tangle we are working with; t_k will refer to the k^{th} coordinate of the vector.[1] Finally, because in many cases we must go around the tangle more than once to get a closed tangloid curve, it will be convenient to have an interpretation for t_k when k is larger than the number of tangle pieces. We will just "cycle" these in the obvious way. For instance, if our tangle is made up of 16 quarter circles, then $t_{18} = t_2$.

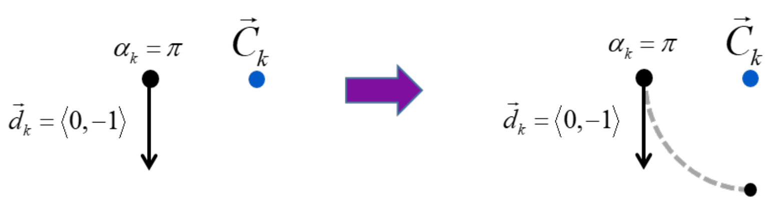

Figure 3: Describing a Tangle Piece

To fully describe the k^{th} tangle piece, it is enough to specify the location of the center $\vec{C}_k$ of the circle , the initial position α_k of the wheel, and the direction of rolling $\vec{d}_k$. (See figure 3.) After making a few key geometric observations, we will describe these parameter values iteratively.

A tangle can "advance" in only a handful of very specific ways. Figure 4 shows four of 16 possibilities.

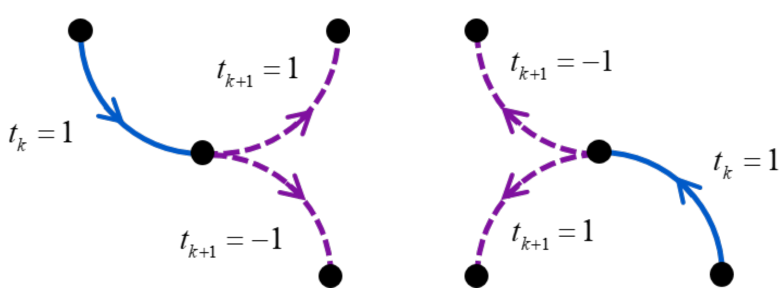

Figure 4: Tangle pieces: what's next?

[1]Interesting identity: given any planar tangle made up of n quarter circles, $\sum_{i=1}^{n} t_i = 4$.

Figure 5 illustrates how the centers, the initial angles, and the direction vectors change depending on the values of t_k and t_{k+1}.

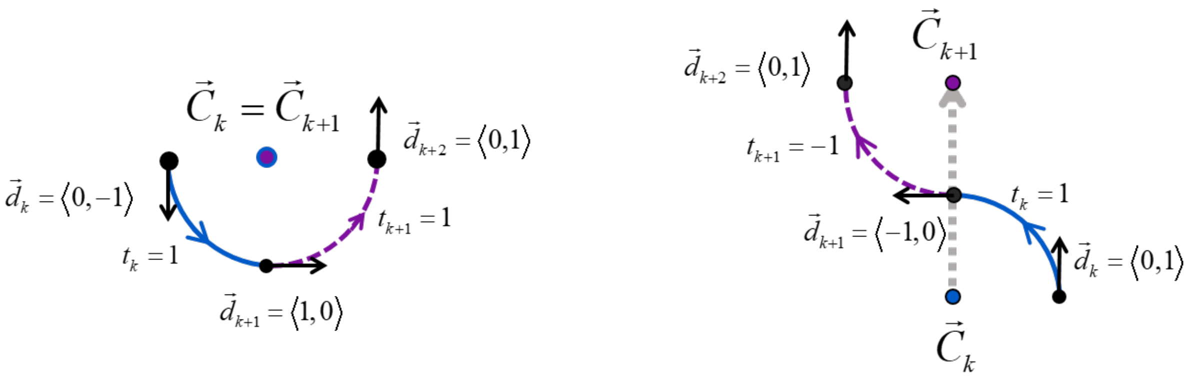

Figure 5: Describing the next tangle piece

If t_k and t_{k+1} are the same, the center will not change. If t_k and t_{k+1} are opposite in sign, the next center will be a distance $2R$ away from the previous; the shift will be either vertical or horizontal. (Referring to figure 5, think about how $\vec{d}_k + \vec{d}_{k+2}$ helps us describe this.) Furthermore, the initial position angle will increase by $\frac{\pi}{2}$ if $t_{k+1} = +1$ and decrease by $\frac{\pi}{2}$ if $t_{k+1} = -1$. Finally, the direction vector will rotate by $\frac{\pi}{2}$ if $t_k = +1$ and by $-\frac{\pi}{2}$ if $t_k = -1$. Thus we will need the following rotation matrices to update the direction vectors:

$$\mathcal{R}_{+1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{R}_{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Description of the tangle given by $\langle t_1, t_2, t_3, \ldots, t_n \rangle$: The initial center is $C_1 = \langle -R, 0 \rangle$, the initial angle is $\alpha_1 = 0$, and the initial (unit) direction vector is $\vec{d}_1 = \langle 0, 1 \rangle$. The remaining points are given iteratively by the following formulas:

$$\begin{aligned} \alpha_{k+1} &= \alpha_k + \frac{\pi}{2} t_{k+1} \\ \vec{d}_{k+1} &= \mathcal{R}_{t_k}\left(\vec{d}_k\right) \\ \vec{C}_{k+1} &= \vec{C}_k + R\left(\vec{d}_k + \vec{d}_{k+2}\right) \end{aligned}$$

The k^{th} tangle piece can then be described parametrically by the following vector equation:

$$\vec{T}_k(s) = \vec{C}_k + R \begin{bmatrix} \cos(\alpha_k + t_k s) \\ \sin(\alpha_k + t_k s) \end{bmatrix} \qquad 0 \le s \le \frac{\pi}{2}.$$

When we plot tangles, we may find it useful to plot the points that show where quarter circles start and end.

$$\vec{P}_k = \vec{C}_k + R \begin{bmatrix} \cos(\alpha_k) \\ \sin(\alpha_k) \end{bmatrix}.$$

On to tangloids! Suppose we start with a tangle $\vec{t}$ of length n, with tangle pieces of radius R, a wheel of radius r, and the pen placed a percentage p from the center of the wheel.[2]

Figure 6 highlights the fact that we will write a formula for the k^{th} tangloid piece as a vector sum that keeps track of three pieces of information:

[2] It is tempting to assume $r < R$ because that is necessary for a physical Spirograph, but the formulas don't require this, and the computer doesn't care if we "color outside the lines." In the same way, we can have $p > 1$; the pen can lie outside the wheel.

- the center of the k^{th} tangle piece
- the position of the center of the wheel relative to the k^{th} tangle piece
- the position of the pen relative to the center of the wheel.

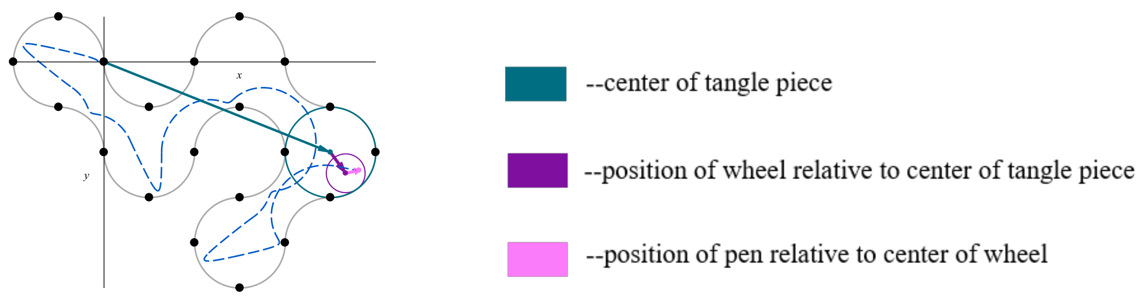

Figure 6: Tangloid formula is a vector Sum

Stipulating that there is a term that we have yet to explain, here is the parametric description for the k^{th} tangloid piece.

$$\overrightarrow{TG}_k(\theta) = \vec{C}_k + (R - t_k\theta)\begin{bmatrix} \cos(t_k\theta + \alpha_k) \\ \sin(t_k\theta + \alpha_k) \end{bmatrix} + pr\begin{bmatrix} \cos\left(t_k\theta - \frac{R}{r}\theta + \phi_k\right) \\ \sin\left(t_k\theta - \frac{R}{r}\theta + \phi_k\right) \end{bmatrix} \qquad 0 \le \theta \le \frac{\pi}{2}.$$

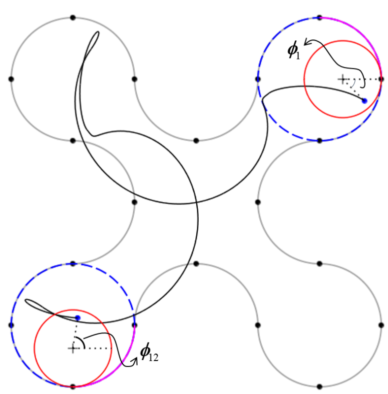

Figure 7: The parameter ϕ

The only term we haven't discussed is ϕ_k. This is an angle that gives the position of the pen, relative to the center of the wheel, at the beginning of the k^{th} tangloid piece. (See figure 7.) We choose the initial position of the pen by designating an angle ϕ_1 relative to the center of the wheel. But the position of the pen at the *beginning* of the $(k+1)^{\text{st}}$ tangloid piece must be the same as the position of the pen at the *end* of the k^{th} tangloid piece; thus $\phi_2, \phi_3, \ldots$ are determined. The parameter ϕ_k is also defined inductively.

An important note: suppose we have a tangloid built on a tangle of length n. The parameters (t_k), $(\vec{C}_k)$, (α_k), and $(\vec{d}_k)$ all "cycle" after the n^{th} term. The sequence (ϕ_k), on the other hand, will not start repeating until the wheel has gone around the tangle sufficiently many times for the tangloid to "close up."

To define the sequence $\phi_1, \phi_2, \phi_3, \ldots$, we need to understand generally how the pen moves as the wheel rolls around the tangle. And more specifically, we need to know how much the position of the pen changes as the wheel rolls along one tangle piece.

Imagine that a wheel rolls counterclockwise through along an arc subtending an angle θ inside a circle, as shown in the left half of figure 8. That is, the wheel rolls a distance of $R\theta$ along the circle. Think of keeping track of the point on the smaller circle where the wheel was originally tangent to the bigger circle. This is the point marked in red. Note that the red point rotates in the direction *opposite* the direction of motion.

That is, while the wheel is rolling counterclockwise around the circle, the red point is rotating clockwise around the wheel.

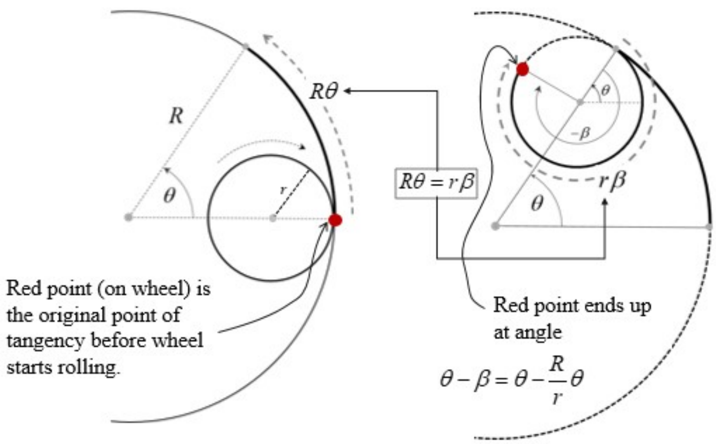

Figure 8: Wheel rolls through an angle θ inside the circle, position of the pen changes by $\theta - \frac{R}{r}\theta$

If the red point rotates by an angle $-\beta$, it has gone a distance of $r\beta$ around the wheel. This is equal to the distance the wheel rolled, so $R\theta = r\beta$. Therefore (calling on parallel lines and corresponding angles) we see that the red point is now at the position $\theta - \beta = \theta - \frac{R}{r}\theta$, as shown in the right half of figure 8. It is easy to see that the position of the pen relative to the wheel center will change by the same angle.

Figure 9 illustrates something similar: suppose the wheel rolls clockwise around the *outside* of the circle through an angle $-\theta$, the red point also rotates clockwise. If it rotates by an angle $-\beta$, the red point is now at a position $-\theta - \beta = -\theta - \frac{R}{r}\theta$.

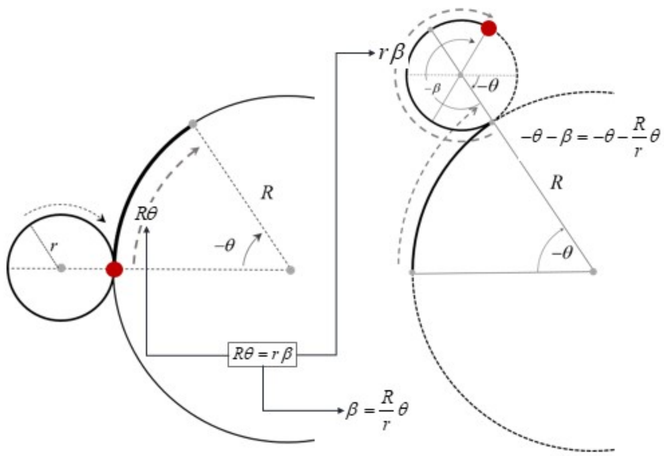

Figure 9: Wheel rolls through an angle θ outside circle, position of the pen changes by $-\theta - \frac{R}{r}\theta$

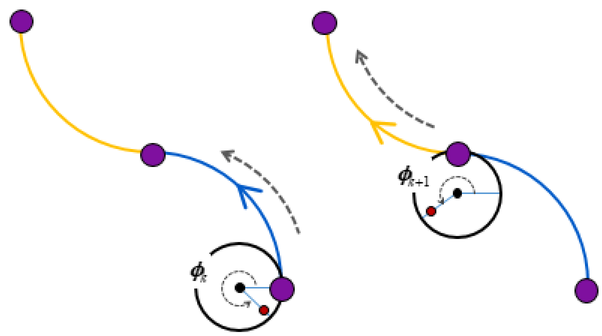

Figure 10. How far does the pen move?

We can now fit the final piece in the tangloid puzzle. Figure 1 and figure 10 suggest that when the wheel comes to a tangle piece labeled with a +1, the pen traces out a hypotrochoid piece, moving counterclockwise inside the corresponding circle. When the wheel comes to a tangle piece labeled with a -1, the pen traces out an epitrochoid piece, moving clockwise outside the corresponding circle. In each of these, the wheel rolls through an angle of $\frac{\pi}{2}$.

Putting this together with the observations made in figures 8 and 9, we can find the relationship between ϕ_k and ϕ_{k+1}. As the wheel rolls along the k^{th} tangle piece, the pen will "advance" by an angle of $\frac{\pi}{2} - \frac{R}{r}\frac{\pi}{2}$ if $t_k = +1$ and by an angle of $-\frac{\pi}{2} - \frac{R}{r}\frac{\pi}{2}$ if $t_k = -1$. Thus

$$\phi_{k+1} = \phi_k + \frac{\pi}{2}t_k - \frac{R}{r}\frac{\pi}{2}.$$

We conclude with the sometimes useful observation that, using back substitution, closed form formulas may be obtained for both α_{k+1} and ϕ_{k+1}:

$$\alpha_{k+1} = \alpha_k + \frac{\pi}{2}t_k \implies \alpha_{k+1} = \alpha_1 + \frac{\pi}{2}\sum_{i=1}^{k} t_i.$$

$$\phi_{k+1} = \phi_k + \frac{\pi}{2}t_k - \frac{R}{r}\frac{\pi}{2} \implies \phi_{k+1} = \phi_1 + \frac{\pi}{2}\sum_{i=1}^{k} t_i - \frac{R}{r}\frac{\pi}{2}k.$$

Summary of Tangloid Building Blocks

Armed with these expressions, you can now program your own tangloids. Here is a summary of what you will need.

- **Inputs**:
  - $\langle t_1, t_2, t_3, \ldots, t_n \rangle$ —sequence of ± 1s that describe the tangle;
  - R — the radius of the tangle pieces;
  - r — the radius of the wheel;
  - p — pr is the distance of the pen from the center of the wheel, so p is a proportion;
  - ϕ_1 — The initial angular position of the pen, relative to the center of the wheel.
- You will also need to be able to access two rotation matrices: The matrix $\mathcal{R}_{+1}$ rotates a vector by $\frac{\pi}{2}$. The matrix $\mathcal{R}_{-1}$ rotates a vector by $-\frac{\pi}{2}$.

$$\mathcal{R}_{+1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{R}_{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- **Iterative formulas for the parameters**:

$$\begin{array}{ll} \alpha_1 = 0 & \alpha_{k+1} = \alpha_k + \frac{\pi}{2} t_{k+1} \\ \vec{d}_1 = \langle 0, 1 \rangle & \vec{d}_{k+1} = \mathcal{R}_{t_k}\left(\vec{d}_k\right) \\ \vec{C}_1 = \langle 0, -R \rangle & \vec{C}_{k+1} = \vec{C}_k + R\left(\vec{d}_k + \vec{d}_{k+2}\right) \\ \phi_1 \text{ (fixed angle, chosen by user)} & \phi_{k+1} = \phi_k + \frac{\pi}{2} t_k - \frac{R}{r}\frac{\pi}{2} \end{array}$$

- Vector formulas for the tangle and the tangloid

$$\begin{aligned} \vec{T}_k(s) &= \vec{C}_k + R \begin{bmatrix} \cos(\alpha_k + t_k s) \\ \sin(\alpha_k + t_k s) \end{bmatrix} \qquad 0 \le s \le \frac{\pi}{2} \text{ and } 1 \le k \le n \\ \overrightarrow{TG}_k(\theta) &= \vec{C}_k + (R - t_k\theta) \begin{bmatrix} \cos(t_k\theta + \alpha_k) \\ \sin(t_k\theta + \alpha_k) \end{bmatrix} + pr \begin{bmatrix} \cos\left(t_k\theta - \frac{R}{r}\theta + \phi_k\right) \\ \sin\left(t_k\theta - \frac{R}{r}\theta + \phi_k\right) \end{bmatrix} \qquad 0 \le \theta \le \frac{\pi}{2} \end{aligned}$$

  The number of tangloid pieces will depend on how many times the wheel must traverse the tangle before the tangloid curve closes up. If we reduce the characteristic ratio $\frac{nR}{4r}$ to lowest terms $\frac{q}{v}$, the number of tangle pieces will be nv.

References

[1] Colbert-Pollack, S., Fisher, M., Schumacher, C. All Tangled Up. *Math Horizons.* 29(November):8-11.