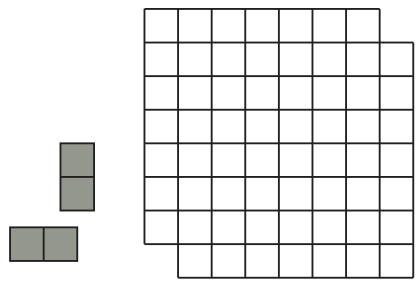
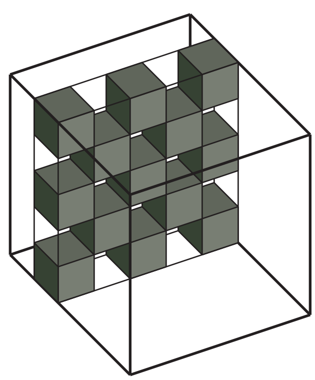
Hint for Problem 213. There is a classic problem in recreational mathematics that asks if it is possible to use 31 dominoes of size 1×2 to cover an 8×8 grid that has a pair of opposite corners removed, as in the figure below. If you have not encountered this problem before, do not read any further until you have tried to figure out the answer on your own!



The key to seeing that a domino tiling of this truncated board is not possible is to color the cells of the board in the standard chessboard manner. Since each 1×2 domino must cover exactly one cell of each color, a board can be tiled by dominoes only if the number of cells of each color is the same, which is not the case for this board.

For the $5 \times 5 \times 5$ cube problem, color each of the 125 cubical cells with two colors in a three-dimensional "chessboard" fashion, so that cells that meet on square faces have different colors. This means that each of the 15 different 5×5 "slices" within the cube will have exactly one more cell of one color than the other. The figure below shows one such slice of 25 cells, with 13 cells of the same color shaded.



Now consider how each of the different types of blocks – $1 \times 1 \times 1$, $1 \times 2 \times 4$ and $2 \times 2 \times 3$ – could intersect these slices. You may find that the positions of a couple of the $1 \times 1 \times 1$ blocks are essentially forced.