Three-Headed Dragon

You have been captured, along with 99 other mathematicians, by a three-headed dragon.

The dragon decides to give you the following test to determine whether you are worthy of saving or if you should be her dinner: Tomorrow, the dragon will put all 100 of you in a line, facing the same direction, with either a red hat or a white hat. Any person can see the hats of the mathematicians in front of her, but she cannot see the color of her own hat, or any of the hats behind her. In any order, you and the other mathematicians can say either “Red” or “White.” If the color a person says matches the color of her own hat, she will be set free; otherwise, she will become the dragon’s dinner. You are allowed to gather beforehand and devise a strategy. How many mathematicians can you save?

Solution: You can save all but one captive. Assign the values 0 and 1 to white hats and red hats, respectively. Have the final person in line add up the value of all the hats in front of him according to this system. If he gets an even number, he says “White” and if he gets an odd number, he says “Red,” sacrificing himself in the process. Then, the 99^{th} person can count up the hats in front of her in the same way. The number she gets will be equal to the number he got if her hat is white and one less than that number if her hat is red. Since the difference can be at most one, the sum she gets will be of the same parity (odd or even) if her hat is white and of a different parity if her hat is red. Thus, from the sum she calculates and the knowledge given to her by the person behind her, she knows the color of her own hat, which she can say to go free. The 98^{th} person, and every person after that, can keep track of the parity of the total as the mathematicians free themselves, and will be able to determine the color of their hat in the same way as the 99^{th} person did. Thus, everyone, except the 100^{th} person in line, is saved.

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