Birthday Cake Surprises

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"I will share my birthday cake with you," said Amy to her little brother Peter. "It will be cut into five pieces. We will make alternate cuts and alternate choices. Since it is my birthday, I will cut first and choose first."

Peter said, "Why don't we just cut it into five equal pieces? By choosing first, you will get $\frac{3}{5}$ of the cake."

The next day, a year older but not wiser, Amy complained, "You tricked me, you little pip squeak. With my advantage of cutting first, I could have got more than $\frac{3}{5}$ of the cake."

"Oh, no, you can't," said Peter. "I have got it all worked out here."

Peter then showed Amy a page of calculations.

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I have a plan which will give me at least $\frac{2}{5}$. Amy will first cut 1 into x and $1-x$, where $0 \le x \le \frac{1}{2}$. There are three cases.

Case 1. $\frac{2}{5} \le x \le \frac{1}{2}$.
I will cut $1-x$ into x and $1-2x$. Now the three pieces are of sizes $1-2x < x = x$. If Amy does not cut either x, neither will I. I will then be sure of getting x plus a second piece, and $x \ge \frac{2}{5}$. If Amy cuts one of x, I will cut the other x in the same proportions. I will get two pieces which add up to $x \ge \frac{2}{5}$.

Case 2. $\frac{1}{5} \le x < \frac{2}{5}$.
I will cut x into $x - \frac{1}{5}$ and $\frac{1}{5}$. Now the three pieces are of sizes $x - \frac{1}{5} < \frac{1}{5} < 1 - x$. If Amy does not cut $1-x$, I will cut this it in halves. The second smallest piece cannot be less than $\frac{1}{2}(x - \frac{1}{5})$, so I will get at least $\frac{1-x}{2} + \frac{1}{2}(x - \frac{1}{5}) = \frac{2}{5}$. Suppose Amy cuts $1-x$ into y and $1-x-y$, where $0 \le y \le \frac{1-x}{2}$. Then I will cut $1-x-y$ into $\frac{2}{5} - y$ and $\frac{3}{5} - x$. Now $y + (\frac{2}{5} - y) = \frac{2}{5} = (x - \frac{1}{5}) + (\frac{3}{5} - x)$. Thus I will get two pieces which add up to $\frac{2}{5}$.

Case 3. $0 \le x < \frac{1}{5}$.
I will cut $1-x$ into $\frac{1}{5}$ and $\frac{4}{5} - x$. The situation is exactly the same as in Case 2.

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Counting her lucky stars, Amy wondered if she had not been the beneficiary of the perverse generosity of her little brother. Imitating Peter's calculations, she tried to prove that she could always get at least $\frac{3}{5}$ of the cake. The analysis turned out to be a bit more complicated.

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It seems like a good idea for me to cut 1 into $\frac{2}{5}$ and $\frac{3}{5}$. There are two cases.

Case 1. Peter cuts $\frac{2}{5}$ into x and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
I will cut $\frac{3}{5}$ into x and $\frac{3}{5}-x$. Now the four pieces are of sizes $x = x \le \frac{2}{5}-x < \frac{3}{5}-x$. No matter what Peter does, the size of the second largest piece is at most $\frac{2}{5}-x$ and the size of the fourth largest piece is at most x. Hence Peter gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.

Case 2. Peter cuts $\frac{3}{5}$ into x and $\frac{3}{5}-x$, where $0 \le x \le \frac{3}{10}$.
If $0 \le x \le \frac{1}{5}$, I will cut $\frac{2}{5}$ into x and $\frac{2}{5}-x$, and the situation is exactly the same as in Case 1. Hence we may assume that $\frac{1}{5} < x \le \frac{3}{10}$. I will cut $\frac{3}{5}-x$ into $\frac{1}{5}$ and $\frac{2}{5}-x$. Now the four pieces are of sizes $\frac{2}{5}-x < \frac{1}{5} < x < \frac{2}{5}$. There are four subcases.

Subcase 2(a). Peter cuts $\frac{2}{5}$ into y and $\frac{2}{5}-y$, where $0 \le y \le \frac{1}{5}$.
Since $y+(\frac{2}{5}-y)=\frac{2}{5}=x+(\frac{2}{5}-x)$, Peter will get two pieces which add up to $\frac{2}{5}$.

Subcase 2(b). Peter cuts x.
If $\frac{1}{5}$ remains the third largest piece, I get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$. If it becomes the second largest piece, Peter gets at most $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$.

Subcase 2(c). Peter cuts $\frac{1}{5}$ into y and $\frac{1}{5}-y$, where $0 \le y \le \frac{1}{10}$.
Since $\frac{2}{5}-x \ge y$, the second smallest piece is at most $\frac{2}{5}-x$. Hence Peter gets at most $(\frac{2}{5}-x)+x=\frac{2}{5}$.

Subcase 2(d). Peter cuts $\frac{2}{5}-x$.
I get at least $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.

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On his birthday, Peter said to Amy, “I have a surprise for you. We will cut my birthday cake into five pieces like last time.”

“What is the big surprise?” asked Amy. “Won’t you just get $\frac{3}{5}$ of it and I get $\frac{2}{5}$?”

“This is if we stick by your rules. It being my birthday, I will choose first. However, as my Most Esteemed Senior Sister, you are accorded the advantage of cutting first.”

Amy was instantly on guard. However, she could not see any down side, and sealed her fate by agreeing to the deal.

Content with getting $\frac{2}{5}$ of the cake, she began by following the strategy she had worked out earlier, cutting 1 into $\frac{2}{5}$ and $\frac{3}{5}$. Peter surprised her by cutting $\frac{3}{5}$ into $\frac{27}{53}$ and $\frac{24}{265}$, even though she had always known that her little brother would do strange things. She could never understand him.

Anyway, $\frac{24}{265} < \frac{1}{5}$. According to her strategy, she cut $\frac{2}{5}$ into $\frac{82}{265}$ and $\frac{24}{265}$. Peter then cut $\frac{82}{265}$ into $\frac{41}{265}$ and $\frac{41}{265}$, and got $\frac{27}{53}+\frac{41}{265}+\frac{24}{265}=\frac{40}{53}$. Instead of getting $\frac{3}{5}$ of the cake, Peter got more than $\frac{3}{4}$ of it.

Licking her wounds and what was left of her much smaller share of the cake, Amy suddenly realized that under the new rules, her strategy would not guarantee that she would get $\frac{2}{5}$ of the cake. She now reconsidered her last move. At the moment, the cake was in three pieces, of sizes $\frac{27}{53}$, $\frac{2}{5}$ and $\frac{24}{265}$. What else could she have done?

The analysis was painfully straight-forward.

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There are three cases.

Case 1. I cut $\frac{24}{265}$.
Peter will cut $\frac{2}{5}$ into $\frac{1}{5}$ and $\frac{1}{5}$, and gets at least $\frac{27}{53}+\frac{1}{5}+\frac{12}{265}=\frac{40}{53}$.

Case 2. I cut $\frac{2}{5}$ into x and $\frac{2}{5}-x$, where $0 \le x \le \frac{1}{5}$.
If $0 \le x \le \frac{24}{265}$, Peter will cut $\frac{2}{5}-x$ in halves and gets $\frac{27}{53}+\frac{1}{5}-\frac{x}{2}+x \ge \frac{40}{53}$. If $\frac{41}{265} \le x \le \frac{1}{5}$, Peter will cut $\frac{2}{5}-x$ into x and $\frac{2}{5}-2x$ and gets $\frac{27}{53}+x+\frac{2}{5}-2x \ge \frac{40}{53}$. So I should cut $\frac{2}{5}$ into three equal pieces. Then Peter will only get $\frac{27}{53}+\frac{2}{15}+\frac{24}{265}=\frac{583}{795}$. This is less than $\frac{3}{4}$, but still a lot bigger than the $\frac{3}{5}$ I got last time.

Case 3. I cut $\frac{27}{53}$.
As in Case 2, I should cut $\frac{27}{53}$ into three equal pieces. Then Peter will only get $\frac{2}{5}+\frac{9}{53}+\frac{24}{265}=\frac{35}{53}$, which is just under $\frac{2}{3}$. That was probably the best I could have done.

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The next time she was left home alone to baby-sit her little brother, Amy sat on top of Peter and pulverized him.

"You little pip squeak! You tricked me again. Tell me! Could I have done better than getting $\frac{18}{53}$ of your cake?"

"No, and that is why I cut $\frac{3}{5}$ into $\frac{27}{53}$ and $\frac{24}{265}$. If you let me up, I will show you my calculations."

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Amy cuts 1 into $a \le b$. There are four cases.

Case A. $\frac{34}{53} \le a \le 1$ so that $0 \le b \le \frac{19}{53}$.
I cut b into $\frac{b}{2}$ and $\frac{b}{2}$. If Amy does not cut a, I just cut off $\frac{1}{53}$ from another piece, and will get three pieces with total volume at least $a+\frac{1}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \le d$. I cut the d into $\frac{d}{2}$ and $\frac{d}{2}$, and will get three pieces with total volume at least $c+\frac{d}{2}+\frac{b}{2} \ge \frac{a}{4}+\frac{1}{2} \ge \frac{35}{53}$.

Case B. $\frac{33}{53} \le a \le \frac{34}{53}$ so that $\frac{19}{53} \le b \le \frac{20}{53}$.
I cut b into $\frac{18}{53}$ and $b-\frac{18}{53}$. If Amy does not cut a, I just cut off $\frac{2}{53}$ from another piece, and will get three pieces with total volume at least $a+\frac{2}{53} \ge \frac{35}{53}$. Suppose Amy cuts a into $c \le d$. We consider four subcases.

Subcase B1. $\frac{18}{53} \ge c \ge \frac{17}{53} \ge d \ge b-\frac{18}{53}$.
I cut c into d and $c-d$, and will get three pieces with total volume at least $\frac{18}{53}+d+\min\{c-d, b-\frac{18}{53}\}$. In the former instance, it is at least $\frac{18}{53}+c \ge \frac{35}{53}$. In the latter instance, it is at least $d+b=1-c \ge \frac{35}{53}$.

Subcase B2. $\frac{17}{53} \ge c \ge d \ge b-\frac{18}{53}$.
I cut $b-\frac{18}{53}$ into $\frac{b}{2}-\frac{9}{53}$ and $\frac{b}{2}-\frac{9}{53}$, and will get three pieces with total volume at least $\frac{18}{35}+d+\frac{b}{2}-\frac{9}{35}=\frac{62}{53}-\frac{b}{2}-c \ge \frac{35}{53}$.

Subcase B3. $c \ge \frac{18}{53} \ge d \ge b-\frac{18}{53}$.
I cut $\frac{18}{53}$ into d and $\frac{18}{53}-d$, and will get three pieces with total volume at least $c+d+\min\{\frac{18}{35}-d, b-\frac{18}{35}\}$. In the former instance, it is at least $c+\frac{18}{53} \ge \frac{36}{53}$. In the latter instance, it is at least $a+b-\frac{18}{53}=\frac{35}{53}$.

Subcase B4. $c \ge \frac{18}{53} \ge b-\frac{18}{53} \ge d$.
I cut $\frac{18}{53}$ into $\frac{9}{53}$ and $\frac{9}{53}$. Since $d \le b-\frac{18}{53} \le \frac{2}{53}$, $c \ge \frac{31}{53}$. Hence I will get three pieces with total volume at least $c+\frac{9}{53} \ge \frac{40}{53}$.

Case C. $\frac{27}{53} \le a \le \frac{33}{53}$ so that $\frac{20}{53} \le b \le \frac{26}{53}$.
I cut a into $\frac{27}{53}$ and $a - \frac{27}{53}$. If Amy does not cut $\frac{27}{53}$, I just cut off $\frac{8}{53}$ from another piece. If it is the second largest, then Amy gets two pieces with total volume at most $\frac{16}{53}$. Otherwise, I will get three pieces with total volume at least $\frac{27}{53} + \frac{8}{53} = \frac{35}{53}$. Suppose Amy cuts $\frac{27}{53}$ into $c \ge d$. There are four subcases.

Subcase C1. $\frac{27}{106} \le c \le \frac{15}{53}$ so that $\frac{12}{53} \le d \le \frac{27}{106}$.
I cut $a - \frac{27}{53} = \frac{26}{53} - b$ into $\frac{13}{53} - \frac{b}{2}$ and $\frac{13}{53} - \frac{b}{2}$. I will get three pieces with total volume $b + d + (\frac{13}{53} - \frac{b}{2}) \ge \frac{35}{53}$.

Subcase C2. $\frac{15}{53} \le c \le \frac{18}{53}$ so that $\frac{9}{53} \le d \le \frac{12}{53}$.
I cut c into d and $c - d$. I will get three pieces with total volume $b + d + \min\{c - d, a - \frac{27}{53}\}$. In the former instance, it is at least $b + c \ge \frac{35}{53}$. In the latter instance, it is at least $d + 1 - \frac{27}{53} \ge \frac{35}{53}$.

Subcase C3. $\frac{18}{53} \le c \le \frac{24}{53}$ so that $\frac{3}{53} \le d \le \frac{9}{53}$.
I cut c into $\frac{c}{2}$ and $\frac{c}{2}$. Amy gets two pieces with total volume at most $\frac{c}{2} + \max\{d, a - \frac{27}{53}\}$. In the former instance, it is at most $\frac{27}{53} - c \le \frac{18}{35}$. In the latter instance, it is at most $\frac{12}{53} + \frac{6}{53} = \frac{18}{53}$.

Subcase C4. $\frac{24}{53} \le c \le \frac{27}{53}$ so that $0 \le d \le \frac{3}{53}$.
I cut b into $\frac{b}{2}$ and $\frac{b}{2}$. I will get three pieces with total volume at least $c + \frac{n}{2} + \min\{d, a - \frac{27}{53}\}$. In the former instance, it is at least $\frac{27}{53} + \frac{b}{2} \ge \frac{37}{53}$. In the latter instance, it is at least $c + 1 - \frac{27}{53} - \frac{b}{2} \ge \frac{37}{53}$.

Case D. $\frac{1}{2} \le a \le \frac{27}{53}$ so that $\frac{26}{53} \le b \le \frac{1}{2}$.
I pass. Whicever piece Amy now cuts, I cuts off from the larger of the two new pieces a piece equal to $\frac{1}{3}$ of the piece Amy cuts. This piece will be the third largest, and I will get three pieces with total volume at least $b + \frac{a}{3} = 1 - \frac{2a}{3} \ge \frac{35}{53}$.

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Overawed, Amy fell into a brown study while Peter made good his escape. For the next several days, she tried to prove that she could always get at least $\frac{18}{53}$ of the cake. She was justifiably proud when she finally completed her analysis. She was learning fast, in order to stay on top of her little brother.

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I will need the following two preliminary results.

Lemma 1.
Suppose before Peter's final cut, the four pieces have volumes w, x, y and z in non-ascending order. If $x \le 2y$, then I can get two pieces with total volume at least x.

Proof:
If Peter cuts either of the smallest two pieces, the second largest piece will have volume x. There is nothing further to prove. Hence Peter must cut either of the largest two pieces, into two pieces both with volume smaller than x. Because $x \le 2y$, at least one of the new pieces has volume less than y. If the original piece with volume y is still the third largest, then the largest has volume at most w and the smallest has volume at most z. Hence Peter gets three pieces with total volume at most $w + y + z$, so that I will get two pieces with total volume at least $1 - w - y - z = x$. On the other hand, if the piece with volume y is now the second largest, then the volume of each of the two new pieces lies between y and $x - y$. Thus the second smallest piece has volume at least $x - y$, and my two pieces will have total volume at least $y + (x - y) = x$.

Lemma 2.
Suppose before Peter's final cut, the four pieces have volumes $w,\ x,\ y$ and z in non-ascending order. If $x \geq 2z$, then I can get two pieces with total volume at least $\min\{y+z, x+\frac{z}{2}\}$.

Proof:
If Peter cuts either of the smallest two pieces, the second smallest piece will have volume at least $\frac{z}{2}$ while the second largest piece will have volume x. Hence I will get two pieces with total volume at least $x+\frac{z}{2}$. If Peter cuts either of the largest two pieces into two pieces both with volume smaller than x, not both can have volume smaller than z since $x \geq 2z$. Hence the second smallest piece has volume at least z while the second largest piece has volume at least y. Hence I will get two pieces with total volume at least $y+z$.

Now my analysis begins. I will cut 1 into $\frac{20}{53}$ and $\frac{33}{53}$. There are two cases.

Case A. Peter cuts $\frac{20}{33}$ into $a \geq b$.
There are three subcases.

Subcase A1. $\frac{18}{53} \leq a \leq \frac{20}{53}$ so that $0 \leq b \leq \frac{2}{53}$.
I cut $\frac{33}{53}$ into $\frac{18}{53}$ and $\frac{15}{53}$. In Lemma 1, $w=a,\ x=\frac{18}{53},\ y=\frac{15}{53}$ and $z=b$, with $x \leq 2y$. Hence I will get two pieces with total volume at least $\frac{18}{53}$.

Subcase A2. $\frac{17}{53} \leq a \leq \frac{18}{53}$ so that $\frac{2}{53} \leq b \leq \frac{3}{53}$.
I cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. In Lemma 2, $w=a,\ x=\frac{17}{53},\ y=\frac{16}{53}$ and $z=b$, with $x \geq 2z$. Now $y+z=\frac{16}{53}+b \geq \frac{18}{53}$ while $x+\frac{z}{2}=\frac{17}{53}+\frac{b}{2} \geq \frac{18}{53}$. Either way, I will get two pieces with total volume $\frac{18}{53}$.

Subcase A3. $\frac{10}{53} \leq a \leq \frac{17}{53}$ so that $\frac{3}{53} \leq b \leq \frac{10}{53}$.
I still cut $\frac{33}{53}$ into $\frac{17}{53}$ and $\frac{16}{53}$. The total volume of the smallest four pieces is $1-\frac{17}{53}=\frac{36}{53}$. I am guaranteed to get at least half of that, which is $\frac{18}{53}$.

Case B. Peter cuts $\frac{33}{53}$ into $a \geq b$.
There are seven subcases.

Subcase B1. $\frac{27}{53} \leq a \leq \frac{33}{53}$, so that $0 \leq b \leq \frac{6}{53}$.
I cut a into $\frac{18}{53}$ and $a-\frac{18}{53}$. In Lemma 1, $w=\frac{20}{53},\ x=\frac{18}{53},\ y=a-\frac{18}{53}$ and $z=b$, with $x \leq 2y$. Hence I will get two pieces with total volume at least $\frac{18}{53}$.

Subcase B2. $\frac{51}{106} \leq a \leq \frac{27}{53}$, so that $\frac{6}{53} \leq b \leq \frac{15}{106}$.
I cut a into $\frac{15}{53}$ and $a-\frac{15}{53}$. In Lemma 2, $w=\frac{20}{53}, x=\frac{15}{53},\ y=a-\frac{15}{53}$ and $z=b$, with $x \geq 2z$. Now $y+z=a+b-\frac{15}{53}=\frac{33}{53}-\frac{15}{53}=\frac{18}{53}$ while $x+\frac{z}{2}=\frac{15}{53}+\frac{b}{2} \geq \frac{18}{53}$. Either way, I will get two pieces with total volume at least $\frac{18}{53}$.

Subcase B3. $\frac{25}{53} \leq a \leq \frac{51}{106}$, so that $\frac{15}{106} \leq b \leq \frac{8}{53}$.
I cut a into $a-b-\frac{3}{53}$ and $b+\frac{3}{53}$. If Peter cuts either $b+\frac{3}{53}$ or b, the second smallest piece is at least $\frac{b}{2}$, and I will get two pieces with total volume at least $a-b-\frac{3}{53}+\frac{b}{2}=a+b-\frac{3}{53}-\frac{3b}{2} \geq \frac{18}{53}$. Suppose Peter cuts either $\frac{20}{53}$ or $a-b-\frac{3}{53}$. If both new pieces are less than b, then I will get two pieces with total volume at least $b+\frac{3}{53}+\frac{1}{2}(a-b-\frac{3}{53})=\frac{18}{53}$. If at least one of the new pieces is greater than b, then the second largest piece is at least $b+\frac{3}{53}$ so that I will get two pieces with total volume at least $b+\frac{3}{53}+b \geq \frac{18}{53}$.

Subcase B4. $\frac{23}{53} \leq a \leq \frac{25}{53}$, so that $\frac{8}{53} \leq b \leq \frac{10}{53}$.
I pass. If Peter then cuts b, I will get at least $\frac{20}{53}$. Hence he must cut a or $\frac{20}{53}$. After Peter's final cut, if b is still the third largest, then Peter gets three pieces with total volume at most $a + b + 0 = \frac{33}{53}$, so that I will get two pieces with total volume at least $\frac{18}{53}$. If b becomes the second smallest, then the second largest is at least $\frac{10}{53}$, and I will get two pieces with total volume at least $\frac{10}{53} + b \geq \frac{18}{53}$.

Subcase B5. $\frac{20}{53} \leq a \leq \frac{23}{53}$, so that $\frac{10}{53} \leq b \leq \frac{13}{53}$.
I also pass. In Lemma 1, $w = a,\ x = \frac{20}{53},\ y = b$ and $z = 0$, with $x \leq 2y$. Hence I will get two pieces with total volume at least $\frac{20}{53}$.

Subcase B6. $\frac{18}{53} \leq a \leq \frac{20}{53}$, so that $\frac{13}{53} \leq b \leq \frac{15}{53}$.
I still pass. In Lemma 1, $w = \frac{20}{53},\ x = a,\ y = b$ amd $z = 0$, with $x \leq 2y$. Hence I will get two pieces with total volume at least $\frac{18}{53}$.

Subcase B7. $\frac{33}{106} \leq a \leq \frac{18}{53}$, so that $\frac{15}{53} \leq b \leq \frac{33}{106}$.
I cut $\frac{20}{53}$ into $\frac{14}{53}$ and $\frac{6}{53}$. In Lemma 2, $w = a,\ x = b, y = \frac{14}{53}$ and $z = \frac{6}{53}$ with $x \geq 2z$. Now $y + z = \frac{20}{53}$ while $x + \frac{z}{2} = b + \frac{3}{53} \geq \frac{18}{53}$. Either way, I will get two pieces with total volume at least $\frac{18}{53}$.