A1. Determine all possible values of the expression
\[ A^3 + B^3 + C^3 - 3ABC \]
where \( A, B, \) and \( C \) are nonnegative integers.

**Answer.** The possible values are the nonnegative integers that are either divisible by 9 or not divisible by 3.

**Solution 1.** Let \( f(A, B, C) = A^3 + B^3 + C^3 - 3ABC \). By direct computation, for nonnegative integers \( A \) we have \( f(A, A, A + 1) = 3A + 1 \) and \( f(A, A, A) = 0 \), while for positive integers \( A \) we have \( f(A, A, A - 1) = 3A - 1 \) and \( f(A, A + 1, A - 1) = 9A \). This shows that all the values listed in the answer can actually be obtained. To show that no other values are possible, we show that of the two factors \( A, B, C \) we have
\[ \frac{1}{3}(A^3 + B^3 + C^3) \geq \sqrt[3]{A^3 B^3 C^3} = ABC, \]
and therefore \( f(A, B, C) \geq 0 \).

So when \( A, B, C \) are nonnegative integers, the value \( f(A, B, C) \) must be a nonnegative integer; it remains to show that if \( f(A, B, C) \) is divisible by 3, then it is also divisible by 9. Note that \( f(A, B, C) \equiv A^3 + B^3 + C^3 \equiv A + B + C \pmod{3} \), so we are concerned with the case that \( A + B + C \equiv 0 \pmod{3} \). In this case we have \( C = 3k - A - B \) for some integer \( k \), and then
\[ f(A, B, C) = A^3 + B^3 + (3k - A - B)^3 - 3AB(3k - A - B) \]
\[ = 9k(a^2 + ab + b^2 - 3k(a + b) + k^2) \]
is divisible by 9, completing the proof.

**Solution 2.** Start by observing the factorization
\[ A^3 + B^3 + C^3 - 3ABC = (A + B + C)(A^2 + B^2 + C^2 - AB - BC - CA) \]
\[ = (A + B + C) \cdot \frac{(A - B)^2 + (B - C)^2 + (C - A)^2}{2}, \]
from which it is clear that the only possible values are nonnegative integers. For \( (A, B, C) = (k, k, k) \) we get 0; for \( (A, B, C) = (k, k, k + 1) \) we get \((3k + 1) \cdot 1\); for \( (A, B, C) = (k, k + 1, k + 1) \) we get \((3k + 2) \cdot 1\); and for \( (A, B, C) = (k, k + 1, k + 2) \) we get \((3k + 3) \cdot 3 = 9(k + 1)\), showing that all the values listed in the answer can actually be obtained. To show that no others are possible, we show that of the two factors \( A + B + C \) and \( \frac{(A - B)^2 + (B - C)^2 + (C - A)^2}{2} \), either both are divisible by 3 or neither is divisible by 3. If \( A + B + C \) is divisible by 3, then either \( A, B, C \) are all equal mod 3, in which case the second factor is clearly divisible by 3, or \( A, B, C \) are all different mod 3, in which case \( (A - B)^2 + (B - C)^2 + (C - A)^2 \) is equal to \( 1 + 1 + 1 \) modulo 3 and the second factor is again divisible by 3. If \( A + B + C \) is not divisible by 3, then \( A, B, C \) take on precisely two different values mod 3 (one twice, the other once), so \( (A - B)^2 + (B - C)^2 + (C - A)^2 \) is equal to \( 0 + 1 + 1 \) modulo 3 and the second factor is not divisible by 3.

A2. In the triangle \( \Delta ABC \), let \( G \) be the centroid, and let \( I \) be the center of the inscribed circle. Let \( \alpha \) and \( \beta \) be the angles at the vertices \( A \) and \( B \), respectively. Suppose that the segment \( IG \) is parallel to \( AB \) and that \( \beta = 2\tan^{-1}(1/3) \). Find \( \alpha \).

**Answer.** \( \alpha = \frac{\pi}{2} \).
Solution 1. Recall that $I$ is the intersection point of the angle bisectors, and $G$ is the intersection point of the medians, of the triangle. Let $Z$ be the “foot” of the angle bisector at $C$ (the intersection of that bisector and $AB$) and $M$ be the midpoint of $AB$. Then because $IG$ and $AB$ are parallel, $\triangle CIG$ and $\triangle CZM$ are similar triangles, so

$$\frac{CI}{IZ} = \frac{CG}{GM} = 2.$$ 

On the other hand, $AI$ is the angle bisector through $A$ in $\triangle ACZ$, which divides the opposite side in the ratio of the sides adjacent to $A$, so $AC = 2 AZ$. Similarly, $BC = 2 BZ$, and adding these two equations we get $AC + BC = 2 AB$. Now use the law of sines to express all the side lengths of $\triangle ABC$ in terms of $BC$:

$$AC = BC \frac{\sin 2 \alpha}{\sin \alpha}, \ AB = BC \frac{\sin 2 \alpha}{\sin \alpha} = BC \frac{\sin(\alpha + \beta)}{\sin \alpha},$$

and it follows that

$$\sin \beta + \sin \alpha = 2 \sin(\alpha + \beta).$$

Given that $\beta = 2 \tan^{-1}(1/3)$, we have

$$\sin 2 \alpha = 2 \sin \alpha \cos 2 \alpha = 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} = \frac{3}{5},$$

$$\cos 2 \alpha = 2 \cos^2 \alpha - 1 = \frac{4}{5},$$

and using the addition formula for sine we get

$$\frac{3}{5} + \sin \alpha = 2 \sin \alpha \cos \beta + 2 \cos \alpha \sin \beta$$

$$= \frac{8}{5} \sin \alpha + \frac{6}{5} \cos \alpha,$$

which simplifies to

$$\sin \alpha + 2 \cos \alpha = 1.$$ 

If we set $x = \cos \alpha, \ y = \sin \alpha$ we have $2x + y = 1, \ x^2 + y^2 = 1$. Eliminating $y$ yields $5x^2 - 4x = 0$, so $x = 0$ or $x = 4/5$, but $x = 4/5$ would yield a negative value for $y$, which is impossible. So $\cos \alpha = 0$ and $\alpha = \pi/2$.

Solution 2. Let $h$ be the length of the altitude from $C$ in $\triangle ABC$, and let $r$ be the radius of the inscribed circle. Then the area of $\triangle ABC$ is equal to $h \cdot AB/2$ and also, as can be seen by dissecting $\triangle ABC$ into three triangles with a common vertex at $I$, equal to $r \cdot (AB + BC + AC)/2$. On the other hand, because $IG$ is parallel to $AB$, the distances from $I$ and from $G$ to $AB$ are equal. The distance from $I$ is $r$, and because the centroid is two-thirds of the way (along the median) from $C$ to $AB$, the distance from $G$ is $h/3$. So $r = h/3$, and comparing the expressions above for the area of the triangle, we see that

$$\frac{1}{3}(AB + BC + AC) = AB, \text{ that is, } BC + AC = 2 AB.$$ 

From here we can proceed as in the first solution, or we can use the law of cosines:

$$AC^2 = AB^2 + BC^2 - 2 AB \cdot BC \cos \beta = AB^2 + BC^2 - \frac{8}{5} AB \cdot BC,$$

so

$$(2AB - BC)^2 = AB^2 + BC^2 - \frac{8}{5} AB \cdot BC,$$ 

which yields $AB = \frac{4}{5} BC, \ AC = \frac{3}{5} BC.$
So \( AB^2 + AC^2 = BC^2 \), and \( \triangle ABC \) is a right triangle with the right angle \( \alpha \) at \( A \).

**A3.** Given real numbers \( b_0, b_1, \ldots, b_{2019} \) with \( b_{2019} \neq 0 \), let \( z_1, z_2, \ldots, z_{2019} \) be the roots in the complex plane of the polynomial

\[
P(z) = \sum_{k=0}^{2019} b_k z^k.
\]

Let \( \mu = (|z_1| + \cdots + |z_{2019}|)/2019 \) be the average of the distances from \( z_1, z_2, \ldots, z_{2019} \) to the origin. Determine the largest constant \( M \) such that \( \mu \geq M \) for all choices of \( b_0, b_1, \ldots, b_{2019} \) that satisfy

\[
1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.
\]

**Answer.** \( M = \left( \frac{1}{2019} \right)^{\frac{1}{2019}} = \frac{1}{\sqrt[2019]{2019}} \).

**Solution.** Because the polynomial factors as

\[
P(z) = b_{2019}(z - z_1)(z - z_2) \cdots (z - z_{2019}),
\]

its constant term is

\[
b_0 = -b_{2019}z_1z_2 \cdots z_{2019},
\]

and therefore we have

\[
|z_1||z_2| \cdots |z_{2019}| = \frac{b_0}{b_{2019}} \geq \frac{1}{2019}.
\]

By the AM-GM inequality,

\[
\mu = (|z_1| + \cdots + |z_{2019}|)/2019 \geq (|z_1||z_2| \cdots |z_{2019}|)^{\frac{1}{2019}} \geq \left( \frac{1}{2019} \right)^{\frac{1}{2019}}.
\]

So to finish the proof, it is enough to exhibit a specific polynomial for which

\[
\mu = M, \text{ where } M = \left( \frac{1}{2019} \right)^{\frac{1}{2019}}. \text{ Note that for such a polynomial, all the } |z_i| \text{ must be equal to } M, \text{ and we must have } b_0 = 1, b_{2019} = 2019. \text{ Specifically, let } \omega = \exp(2\pi i/2020), \text{ a primitive } 2020\text{th root of unity. Then the polynomial}
\]

\[
P(z) = 2019(z - M\omega)(z - M\omega^2) \cdots (z - M\omega^{2019})
\]

\[
= 2019 \cdot \frac{z^{2020} - M^{2020}}{z - M}
\]

\[
= 2019(z^{2019} + Mz^{2018} + \cdots + M^{2018}z + M^{2019})
\]

has coefficients

\[
b_0 = 2019 M^{2019} = 1 < b_1 = 2019 M^{2018} < \cdots < b_{2018} = 2019 M < b_{2019} = 2019
\]

and roots \( z_i = M\omega^i \) with \( |z_i| = M \), so \( \mu = M \) as desired.
A4. Let \( f \) be a continuous real-valued function on \( \mathbb{R}^3 \). Suppose that for every sphere \( S \) of radius 1, the integral of \( f(x, y, z) \) over the surface of \( S \) equals 0. Must \( f(x, y, z) \) be identically 0?

**Answer.** No.

**Solution.** We will show that any nonzero continuous function \( f \) that depends on just one of the three variables \( x, y, z \) and that is periodic with period 2 and average value 0 provides a counterexample. (One such function is \( f(x, y, z) = \sin(\pi z) \).) Let \( f \) be such a function, and assume that \( f \) depends only on \( z \). Then to find the surface integral of \( f(z) \) over a sphere of radius 1, we may assume without loss of generality that the sphere is centered at \((0, 0, c)\); it can then be parametrized by

\[
x = \sin \phi \cos \theta, \quad y = \sin \phi \sin \theta, \quad z = c + \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.
\]

For this parametrization \( R(\phi, \theta) = <\sin \phi \cos \theta, \sin \phi \sin \theta, c + \cos \phi> \) we have the surface area element

\[
dS = |\partial R/\partial \phi \times \partial R/\partial \theta| \, d\phi \, d\theta
= |<\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi>| \, d\phi \, d\theta
= |\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi>| \, d\phi \, d\theta
= \sin \phi \, d\phi \, d\theta.
\]

Therefore, the surface integral is equal to

\[
\int_0^\pi \int_0^{2\pi} f(c + \cos \phi) \sin \phi \, d\theta \, d\phi = 2\pi \int_0^\pi f(c + \cos \phi) \sin \phi \, d\phi
= 2\pi \int_{c-1}^{c+1} f(z) \, dz.
\]

Because the integral is over a full period of \( f \) and \( f \) has average value 0, the integral is always 0.

A5. Let \( p \) be an odd prime number, and let \( \mathbb{F}_p \) denote the field of integers modulo \( p \). Let \( \mathbb{F}_p[x] \) be the ring of polynomials over \( \mathbb{F}_p \), and let \( q(x) \in \mathbb{F}_p[x] \) be given by

\[
q(x) = \sum_{k=1}^{p-1} a_k x^k,
\]

where

\[
a_k = k^{(p-1)/2} \mod p.
\]

Find the greatest nonnegative integer \( n \) such that \((x - 1)^n\) divides \( q(x) \) in \( \mathbb{F}_p[x] \).

**Answer.** \( n = \frac{p-1}{2} \).

**Solution.** Let \( m = \frac{p-1}{2} \). Then the polynomial \( q(x) \) can be obtained from the polynomial \( P(x) = \sum_{k=0}^{p-1} x^k \) by \( m \) applications of the linear operator \( L \) on \( \mathbb{F}_p[x] \) defined by \( L(f(x)) = xf'(x) \), where the prime denotes taking the formal derivative. Note that if a polynomial \( f(x) \in \mathbb{F}_p[x] \) is divisible by \((x - 1)^r\), then \( L(f(x)) \) is divisible by \((x - 1)^{r-1}\) (to see this, write \( f(x) = (x - 1)^r g(x) \) and differentiate both sides). We
Let throughout this solution, we will abbreviate \( \lim_{x \to 0^+} \), and similarly for \( \lim_{x \to 0^+} \) and \( \lim_{x \to 0^-} \). Suppose that under the assumptions of the problem, \( \lim g'(x) \neq 0 \). We will show that then \( \lim_{x \to 0^+} x^r |g''(x)| = \infty \). First of all, from the given limit of \( g(x)/x^r \) we have \( g(0) = 0 \). We claim that
\[
\lim_{x \to 0^+} g'(x) \leq 0 \leq \lim_{x \to 0^+} \sup g'(x). \quad (*)
\]
In fact, if the left-hand inequality would fail, there would be some positive constant \( c \) and some interval \( (0, \delta) \) on which \( g'(x) > c \). But that would imply \( g(x) > cx \) on this interval, in contradiction with the given limit. The argument for the right-hand inequality is similar. Also, if both inequalities were equalities, then we would have \( \lim g'(x) = 0 \) after all, so at least one of the two inequalities in (*) must be strict.

Now suppose that the inequality \( \lim_{x \to 0^+} g'(x) > 0 \) is strict. (The other case is completely analogous.) Choose a positive constant \( C < \lim_{x \to 0^+} g'(x) \). Then because \( g' \) is continuous on \( (0, 1) \) and \( \lim_{x \to 0^+} g'(x) \leq 0 \), we can find a sequence \( (b_n) \) of real numbers tending to zero such that \( g'(b_n) = C \), and for each \( b_n \) we can find a number \( a_n \) with \( 0 < a_n < b_n \) such that \( g'(a_n) = C/2 \) and such that on the interval \( [a_n, b_n] \), \( g'(x) \geq C/2 \). By the Mean Value Theorem on that interval, we then have \( g(b_n) - g(a_n) \geq C(b_n - a_n)/2 \), so at least one of \( |g(b_n)| \geq C(b_n - a_n)/4 \) and
\[ |g(a_n)| \geq C(b_n - a_n)/4 \] must hold. Let

\[ t_n = \frac{b_n - a_n}{b_n^r}. \]

Then because we know \( \lim g(x)/x^r = 0 \), it follows that \( t_n \to 0 \) as \( n \to \infty \), and in particular, \( a_n/b_n \) approaches 1 as \( n \to \infty \).

Finally, applying the Mean Value Theorem to \( g' \) on the interval \([a_n, b_n]\) shows the existence of \( u_n \in (a_n, b_n) \) such that

\[ g''(u_n) = \frac{g'(b_n) - g'(a_n)}{b_n - a_n} = \frac{C - C/2}{b_n - a_n} = \frac{C}{2t_n b_n^r}. \]

If we take \( n \) sufficiently large so that \( a_n/b_n \geq (2/3)^{1/r} \), we will have \( 3u_n^r > 3a_n^r \geq 2b_n^r \) and thus

\[ g''(u_n) \geq \frac{C}{3t_n u_n^r}, \quad \text{that is,} \quad u_n^r g''(u_n) \geq \frac{C}{3t_n}. \]

As \( t_n \) approaches 0 as \( n \to \infty \), it follows that \( \lim \sup x^r g''(x) = \infty \).