A1. Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are nonnegative integers.

Answer. The possible values are the nonnegative integers that are either divisible by 9 or *not* divisible by 3.

Solution 1. Let $f(A, B, C) = A^3 + B^3 + C^3 - 3ABC$. By direct computation, for nonnegative integers A we have $f(A, A, A+1) = 3A + 1$ and $f(A, A, A) = 0$, while for positive integers A we have $f(A, A, A-1) = 3A - 1$ and $f(A, A+1, A-1) = 9A$. This shows that all the values listed in the answer can actually be obtained. To show that no other values are possible, first note that by the AM-GM inequality, for nonnegative A, B, C we have

$$\frac{1}{3}(A^3 + B^3 + C^3) \geq \sqrt[3]{A^3B^3C^3} = ABC, \text{ and therefore } f(A, B, C) \geq 0.$$

So when A, B, C are nonnegative integers, the value $f(A, B, C)$ must be a nonnegative integer; it remains to show that if $f(A, B, C)$ is divisible by 3, then it is also divisible by 9. Note that $f(A, B, C) \equiv A^3 + B^3 + C^3 \equiv A + B + C \pmod 3$, so we are concerned with the case that $A + B + C \equiv 0 \pmod 3$. In this case we have $C = 3k - A - B$ for some integer k, and then

$$\begin{aligned} f(A, B, C) &= A^3 + B^3 + (3k - A - B)^3 - 3AB(3k - A - B) \\ &= 9k(a^2 + ab + b^2 - 3k(a + b) + 3k^2) \end{aligned}$$

is divisible by 9, completing the proof.

Solution 2. Start by observing the factorization

$$\begin{aligned} A^3 + B^3 + C^3 - 3ABC &= (A + B + C)(A^2 + B^2 + C^2 - AB - BC - CA) \\ &= (A + B + C) \cdot \frac{(A - B)^2 + (B - C)^2 + (C - A)^2}{2}, \end{aligned}$$

from which it is clear that the only possible values are nonnegative integers. For $(A, B, C) = (k, k, k)$ we get 0; for $(A, B, C) = (k, k, k + 1)$ we get $(3k + 1) \cdot 1$; for $(A, B, C) = (k, k + 1, k + 1)$ we get $(3k + 2) \cdot 1$; and for $(A, B, C) = (k, k + 1, k + 2)$ we get $(3k + 3) \cdot 3 = 9(k + 1)$, showing that all the values listed in the answer can actually be obtained. To show that no others are possible, we show that of the two factors $A + B + C$ and $\dfrac{(A - B)^2 + (B - C)^2 + (C - A)^2}{2}$, either both are divisible by 3 or neither is divisible by 3. If $A + B + C$ is divisible by 3, then either A, B, C are all equal mod 3, in which case the second factor is clearly divisible by 3, or A, B, C are all different mod 3, in which case $(A - B)^2 + (B - C)^2 + (C - A)^2$ is equal to $1 + 1 + 1$ modulo 3 and the second factor is again divisible by 3. If $A + B + C$ is not divisible by 3, then A, B, C take on precisely two different values mod 3 (one twice, the other once), so $(A - B)^2 + (B - C)^2 + (C - A)^2$ is equal to $0 + 1 + 1$ modulo 3 and the second factor is not divisible by 3.

A2. In the triangle ΔABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

Answer. $\alpha = \dfrac{\pi}{2}$.

Solution 1. Recall that I is the intersection point of the angle bisectors, and G is the intersection point of the medians, of the triangle. Let Z be the "foot" of the angle bisector at C (the intersection of that bisector and AB) and M be the midpoint of AB. Then because IG and AB are parallel, ΔCIG and ΔCZM are similar triangles, so

$$\frac{CI}{IZ} = \frac{CG}{GM} = 2.$$

On the other hand, AI is the angle bisector through A in ΔACZ, which divides the opposite side in the ratio of the sides adjacent to A, so $AC = 2\,AZ$. Similarly, $BC = 2\,BZ$, and adding these two equations we get $AC + BC = 2\,AB$. Now use the law of sines to express all the side lengths of ΔABC in terms of BC:

$$AC = BC\,\frac{\sin\beta}{\sin\alpha}\,,\;\; AB = BC\,\frac{\sin(\pi-\alpha-\beta)}{\sin\alpha} = BC\,\frac{\sin(\alpha+\beta)}{\sin\alpha},$$

and it follows that

$$\sin\beta + \sin\alpha = 2\,\sin(\alpha+\beta)\,.$$

Given that $\beta = 2\tan^{-1}(1/3)$, we have

$$\sin\beta = 2\sin(\tan^{-1}(1/3))\cos(\tan^{-1}(1/3)) = 2\cdot\frac{1}{\sqrt{10}}\cdot\frac{3}{\sqrt{10}} = \frac{3}{5}\,,$$

$$\cos\beta = 2\cos^2(\tan^{-1}(1/3)) - 1 = \frac{4}{5}\,,$$

and using the addition formula for sine we get

$$\begin{aligned}\frac{3}{5} + \sin\alpha &= 2\,\sin\alpha\,\cos\beta + 2\,\cos\alpha\,\sin\beta \\ &= \frac{8}{5}\sin\alpha + \frac{6}{5}\cos\alpha,\end{aligned}$$

which simplifies to

$$\sin\alpha + 2\cos\alpha = 1\,.$$

If we set $x = \cos\alpha,\; y = \sin\alpha$ we have $2x + y = 1$, $x^2 + y^2 = 1$. Eliminating y yields $5x^2 - 4x = 0$, so $x = 0$ or $x = 4/5$, but $x = 4/5$ would yield a negative value for y, which is impossible. So $\cos\alpha = 0$ and $\alpha = \pi/2$.

Solution 2. Let h be the length of the altitude from C in ΔABC, and let r be the radius of the inscribed circle. Then the area of ΔABC is equal to $h \cdot AB/2$ and also, as can be seen by dissecting ΔABC into three triangles with a common vertex at I, equal to $r \cdot (AB + BC + AC)/2$. On the other hand, because IG is parallel to AB, the distances from I and from G to AB are equal. The distance from I is r, and because the centroid is two-thirds of the way (along the median) from C to AB, the distance from G is $h/3$. So $r = h/3$, and comparing the expressions above for the area of the triangle, we see that

$$\frac{1}{3}(AB + BC + AC) = AB,\; \text{that is,}\; BC + AC = 2AB\,.$$

From here we can proceed as in the first solution, or we can use the law of cosines:

$$AC^2 = AB^2 + BC^2 - 2\,AB\cdot BC\,\cos\beta = AB^2 + BC^2 - \frac{8}{5}\,AB\cdot BC\,,\;\text{so}$$

$$(2AB - BC)^2 = AB^2 + BC^2 - \frac{8}{5}\,AB\cdot BC\,,\;\text{which yields}\; AB = \frac{4}{5}BC,\; AC = \frac{3}{5}BC\,.$$

So $AB^2 + AC^2 = BC^2$, and ΔABC is a right triangle with the right angle α at A.

A3. Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019\,.$$

Answer. $M = \left(\dfrac{1}{2019}\right)^{\frac{1}{2019}} = \dfrac{1}{\sqrt[2019]{2019}}$.

Solution. Because the polynomial factors as

$$P(z) = b_{2019}(z - z_1)(z - z_2) \cdots (z - z_{2019}),$$

its constant term is

$$b_0 = -b_{2019} z_1 z_2 \cdots z_{2019}\,,$$

and therefore we have

$$|z_1||z_2| \cdots |z_{2019}| = \frac{b_0}{b_{2019}} \geq \frac{1}{2019}.$$

By the AM-GM inequality,

$$\mu = (|z_1| + \cdots + |z_{2019}|)/2019 \geq (|z_1||z_2| \cdots |z_{2019}|)^{\frac{1}{2019}} \geq \left(\frac{1}{2019}\right)^{\frac{1}{2019}}.$$

So to finish the proof, it is enough to exhibit a specific polynomial for which $\mu = M$, where $M = \left(\dfrac{1}{2019}\right)^{\frac{1}{2019}}$. Note that for such a polynomial, all the $|z_i|$ must be equal to M, and we must have $b_0 = 1, b_{2019} = 2019$. Specifically, let $\omega = \exp(2\pi i/2020)$, a primitive 2020th root of unity. Then the polynomial

$$\begin{aligned} P(z) &= 2019(z - M\omega)(z - M\omega^2) \cdots (z - M\omega^{2019}) \\ &= 2019 \cdot \frac{z^{2020} - M^{2020}}{z - M} \\ &= 2019(z^{2019} + Mz^{2018} + \cdots + M^{2018}z + M^{2019}) \end{aligned}$$

has coefficients

$$b_0 = 2019M^{2019} = 1 < b_1 = 2019M^{2018} < \cdots < b_{2018} = 2019M < b_{2019} = 2019$$

and roots $z_i = M\omega^i$ with $|z_i| = M$, so $\mu = M$ as desired.

A4. Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

Answer. No.

Solution. We will show that any nonzero continuous function f that depends on just one of the three variables x, y, z and that is periodic with period 2 and average value 0 provides a counterexample. (One such function is $f(x,y,z) = \sin(\pi z)$.) Let f be such a function, and assume that f depends only on z. Then to find the surface integral of $f(z)$ over a sphere of radius 1, we may assume without loss of generality that the sphere is centered at $(0,0,c)$; it can then be parametrized by

$$x = \sin\phi\cos\theta,\ y = \sin\phi\sin\theta,\ z = c + \cos\phi,\ \ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

For this parametrization $\mathbf{R}(\phi,\theta) = <\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ c+\cos\phi>$ we have the surface area element

$$\begin{aligned} dS &= |\partial\mathbf{R}/\partial\phi \times \partial\mathbf{R}/\partial\theta|\, d\phi\, d\theta \\ &= |<\cos\phi\cos\theta,\ \cos\phi\sin\theta, -\sin\phi> \times < -\sin\phi\sin\theta, \sin\phi\cos\theta, 0>|\, d\phi\, d\theta \\ &= |<\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi>|\, d\phi\, d\theta = \sin\phi\, d\phi\, d\theta. \end{aligned}$$

Therefore, the surface integral is equal to

$$\begin{aligned} \int_0^{\pi}\int_0^{2\pi} f(c+\cos\phi)\sin\phi\, d\theta\, d\phi &= 2\pi\int_0^{\pi} f(c+\cos\phi)\sin\phi\, d\phi \\ &= 2\pi\int_{c-1}^{c+1} f(z)\, dz. \end{aligned}$$

Because the integral is over a full period of f and f has average value 0, the integral is always 0.

A5. Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p\,.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Answer. $n = \dfrac{p-1}{2}$.

Solution. Let $m = \dfrac{p-1}{2}$. Then the polynomial $q(x)$ can be obtained from the polynomial $P(x) = \sum\limits_{k=0}^{p-1} x^k$ by m applications of the linear operator L on $\mathbb{F}_p[x]$ defined by $L(f(x)) = xf'(x)$, where the prime denotes taking the formal derivative. Note that if a polynomial $f(x) \in \mathbb{F}_p[x]$ is divisible by $(x-1)^r$, then $L(f(x))$ is divisible by $(x-1)^{r-1}$ (to see this, write $f(x) = (x-1)^r g(x)$ and differentiate both sides). We now observe that because the coefficients are in $\mathbb{F}_p$, we can write

$$P(x) = \sum_{k=0}^{p-1} x^k = 1 + x + \cdots + x^{p-1} = \frac{1-x^p}{1-x} = \frac{(1-x)^p}{1-x} = (1-x)^{p-1},$$

so $P(x)$ is divisible by exactly $p-1 = 2m$ factors $x-1$, and by the observations above, $q(x) = L^m(P(x))$ is divisible by at least $2m - m = m$ factors $x-1$. To show that $q(x)$ cannot have more than m factors $x-1$ in $\mathbb{F}_p[x]$, note that the result of applying L another m times to the polynomial $q(x)$ is

$$\begin{aligned} L^m(q(x)) &= \sum_{k=1}^{p-1} k^{p-1} x^k \\ &= \sum_{k=1}^{p-1} x^k \quad \text{(by Fermat's Little Theorem)} \\ &= P(x) - 1. \end{aligned}$$

Because $P(x)$ is divisible by $x-1$, $L^m(q(x))$ is not, so $q(x)$ cannot have more than m factors $x-1$, showing that $n = m$.

A6. Let g be a real-valued function that is continuous on the closed interval $[0,1]$ and twice differentiable on the open interval $(0,1)$. Suppose that for some real $r > 1$,

$$\lim_{x\to 0^+} \frac{g(x)}{x^r} = 0\,.$$

Prove that either

$$\lim_{x\to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x\to 0^+} x^r |g''(x)| = \infty\,.$$

Solution. Throughout this solution, we will abbreviate $\lim\limits_{x\to 0^+}$ by $\lim$, and similarly for $\limsup\limits_{x\to 0^+}$ and $\liminf\limits_{x\to 0^+}$. Suppose that under the assumptions of the problem, $\lim g'(x) \neq 0$. We will show that then $\limsup x^r|g''(x)| = \infty$. First of all, from the given limit of $g(x)/x^r$ we have $g(0) = 0$. We claim that

$$\liminf g'(x) \le 0 \le \limsup g'(x). \quad (*)$$

In fact, if the left-hand inequality would fail, there would be some positive constant c and some interval $(0,\delta)$ on which $g'(x) > c$. But that would imply $g(x) > cx$ on this interval, in contradiction with the given limit. The argument for the right-hand inequality is similar. Also, if both inequalities were equalities, then we would have $\lim g'(x) = 0$ after all, so at least one of the two inequalities in $(*)$ must be strict.

Now suppose that the inequality $\limsup g'(x) > 0$ is strict. (The other case is completely analogous.) Choose a positive constant $C < \limsup g'(x)$. Then because g' is continuous on $(0,1)$ and $\liminf g'(x) \le 0$, we can find a sequence (b_n) of real numbers tending to zero such that $g'(b_n) = C$, and for each b_n we can find a number a_n with $0 < a_n < b_n$ such that $g'(a_n) = C/2$ and such that on the interval $[a_n, b_n]$, $g'(x) \ge C/2$. By the Mean Value Theorem on that interval, we then have $g(b_n) - g(a_n) \ge C(b_n - a_n)/2$, so at least one of $|g(b_n)| \ge C(b_n - a_n)/4$ and

$|g(a_n)| \geq C(b_n - a_n)/4$ must hold. Let

$$t_n = \frac{b_n - a_n}{b_n^r}.$$

Then because we know $\lim g(x)/x^r = 0$, it follows that $t_n \to 0$ as $n \to \infty$, and in particular, a_n/b_n approaches 1 as $n \to \infty$.

Finally, applying the Mean Value Theorem to g' on the interval $[a_n, b_n]$ shows the existence of $u_n \in (a_n, b_n)$ such that

$$g''(u_n) = \frac{g'(b_n) - g'(a_n)}{b_n - a_n} = \frac{C - C/2}{b_n - a_n} = \frac{C}{2t_n b_n^r}.$$

If we take n sufficiently large so that $a_n/b_n \geq (2/3)^{1/r}$, we will have $3u_n^r > 3a_n^r \geq 2b_n^r$ and thus

$$g''(u_n) \geq \frac{C}{3t_n u_n^r}, \quad \text{that is, } u_n^r\, g''(u_n) \geq \frac{C}{3t_n}.$$

As t_n approaches 0 as $n \to \infty$, it follows that $\limsup x^r g''(x) = \infty$.