B1. Denote by \( \mathbb{Z}^2 \) the set of all points \((x, y)\) in the plane with integer coordinates. For each integer \( n \geq 0 \), let \( P_n \) be the subset of \( \mathbb{Z}^2 \) consisting of the point \((0, 0)\) together with all points \((x, y)\) such that \( x^2 + y^2 = 2^k \) for some integer \( k \leq n \). Determine, as a function of \( n \), the number of four-point subsets of \( P_n \) whose elements are the vertices of a square.

**Answer.** \( 5n + 1 \).

**Solution.** Let \( S_k \) be the set of all points \((x, y)\) such that \( x^2 + y^2 = 2^k \), so that

\[
P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{n} S_k.
\]

Then \( S_0 = \{(1, 0), (-1, 0), (0, 1), (0, -1)\} \) and \( S_1 = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \).

For \( k \geq 2 \) and any \((x, y) \in S_k\), we have \( x^2 + y^2 \equiv 0 \mod 4 \), so because 0 and 1 are the only squares \( \mod 4 \), \( x \) and \( y \) must both be even. If we put \( x = 2x_1, y = 2y_1 \), then \( 4(x_1^2 + y_1^2) = 2^k \), so \( x_1^2 + y_1^2 = 2^{k-2} \) and \((x_1, y_1) \in S_{k-2}\). It follows by induction on \( k \) that

\[
S_k = \{(2^q, 0), (-2^q, 0), (0, 2^q), (0, -2^q)\} \quad \text{for } k = 2q \text{ even} \quad \text{and}
\]

\[
S_k = \{(2^t, 2^t), (2^t, -2^t), (-2^t, 2^t), (-2^t, -2^t)\} \quad \text{for } k = 2t + 1 \text{ odd}.
\]

Note that for any \( k \), the four points of \( S_k \) form the vertices of a square; also, for any \( q \) there are four squares with one vertex at the origin, two vertices in \( S_{2q} \), and one vertex in \( S_{2q+1} \) (the square with vertices \((0, 0), (2^q, 0), (2^q, 2^q), (0, 2^q)\) and the three squares obtained from it by rotation through \( \pi/2, \pi, 3\pi/2 \) around the origin), and for any \( t \) there are four squares with one vertex at the origin, two vertices in \( S_{2t+1} \), and one vertex in \( S_{2t+2} \) (the square with vertices \((0, 0), (2^t, 2^t), (0, 2^{t+1}), (-2^t, 2^t)\) and the three squares obtained from it by rotation). Thus when we pass from \( P_n \) to \( P_{n+1} \) by including the points in \( S_n \), we get at least five additional squares, whether \( n \) is even or odd. Because there is exactly one four-point subset of \( P_0 \) (namely \( S_0 \)) that gives a square, there will be exactly \( 5n + 1 \) such subsets of \( P_n \), provided that the only squares of which all vertices are in the set

\[
P_\infty = \bigcup_{n=0}^{\infty} P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{\infty} S_k
\]

are the ones we have mentioned so far.

To see that there are no additional such squares, first note that for all \( k \geq 2 \), all points in \( S_k \) have only even coordinates; if we have a square for which each vertex is in \( \{(0, 0)\} \cup \bigcup_{k=2}^{\infty} S_k \), we can scale down all coordinates by a factor 2 and get another square of which all vertices are in \( P_\infty \). Thus it is sufficient to consider squares for which all vertices are in \( P_\infty \) and at least one vertex is in \( S_0 \cup S_1 \).

It is impossible to have just one of the vertices of such a square be in \( S_0 \cup S_1 \), because the square of the side length from that vertex to any other vertex would be 1 or 2 \( \mod 4 \), whereas the square of a side length not involving that vertex would be 0 \( \mod 4 \). By the same argument, if exactly two of the vertices of such a square are in \( S_0 \cup S_1 \), those two must be opposite vertices of the square. And if three or four of the vertices of such a square are in \( S_0 \cup S_1 \), we can choose two such vertices that are
opposite each other. Thus it is enough to analyze squares of which all vertices are in $P_\infty$ and two opposite vertices are in $S_0 \cup S_1$.

If one of the two opposite vertices in $S_0 \cup S_1$ is in $S_0$, up to rotational symmetry we can assume it is $(1,0)$. Then it can be checked by a quick case analysis that the vertex of the square opposite it cannot be in $S_1$; if it is $(-1,0)$, then the vertices of the square are the four points of $S_0$, otherwise it is $(0,1)$ up to reflectional symmetry, and the vertices of the square are $(0,0), (1,0), (1,1), (0,1)$. The final possibility is that the two opposite vertices in $S_0 \cup S_1$ are both in $S_1$, in which case we can assume up to symmetry that they are $(1,1)$ and $(-1,-1)$ (and the vertices of the square are all the points of $S_1$) or $(1,1)$ and $(-1,1)$ (and the vertices of the square are $(0,0), (1,1), (0,2), (-1,1)$). We have now checked that the only possible squares whose vertices are all in $P_n$ are the $5n + 1$ squares found above.

**B2.** For all $n \geq 1$, let
\[
a_n = \frac{\sin((2k-1)\theta_n)}{\cos^2((k-1)\theta_n) \cos^2(k\theta_n)}.
\]
Determine
\[
\lim_{n \to \infty} \frac{a_n}{n^3}.
\]

**Answer.** $\frac{8}{\pi^3}$.

**Solution.** Let $\theta_n = \frac{\pi}{2n}$, and note that $\sin \theta_n \neq 0$. Then we have
\[
a_n = \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n)}{\cos^2((k-1)\theta_n) \cos^2(k\theta_n)}
= \frac{1}{\sin \theta_n} \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n) \sin \theta_n}{\cos^2((k-1)\theta_n) \cos^2(k\theta_n)}
= \frac{1}{\sin \theta_n} \sum_{k=1}^{n-1} \frac{1}{2} \cos((2k-2)\theta_n) - \cos(2k\theta_n)
= \frac{1}{\sin \theta_n} \sum_{k=1}^{n-1} \frac{1}{2} \cos^2((k-1)\theta_n) - \cos^2(k\theta_n)
= \frac{1}{\sin \theta_n} \sum_{k=1}^{n-1} \left( \frac{1}{\cos^2(k\theta_n)} - \frac{1}{\cos^2((k-1)\theta_n)} \right).
\]
We now see that the sum telescopes, and we get
\[
a_n = \frac{1}{\sin \theta_n} \left( \frac{1}{\cos^2((n-1)\theta_n)} - 1 \right).
\]
Because $n\theta_n = \frac{\pi}{2}$, we have $\cos((n-1)\theta_n) = \cos(\frac{\pi}{2} - \theta_n) = \sin \theta_n$, so
\[
a_n = \frac{1}{\sin^3 \theta_n} - \frac{1}{\sin \theta_n}.
\]
Now let \( n \to \infty \). Then \( \theta_n \to 0 \), so

\[
\lim_{n \to \infty} n \sin \theta_n = \lim_{n \to \infty} n \theta_n = \frac{\pi}{2}.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{a_n}{n^3} = \lim_{n \to \infty} \frac{1}{n^2 (n \sin \theta_n)} = \frac{1}{\frac{\pi}{2}^3} - 0 = \frac{8}{\pi^3}.
\]

**B3.** Let \( Q \) be an \( n \)-by-\( n \) real orthogonal matrix, and let \( u \in \mathbb{R}^n \) be a unit column vector (that is, \( u^T u = 1 \)). Let \( P = I - 2u u^T \), where \( I \) is the \( n \)-by-\( n \) identity matrix. Show that if 1 is not an eigenvalue of \( Q \), then 1 is an eigenvalue of \( PQ \).

**Solution.** Note that \( P(u) = u - 2uu^T u = u - 2u = -u \), while if \( v \in \mathbb{R}^n \) is a vector orthogonal to \( u \), that is, if \( u^T v = 0 \), we have \( P(v) = v - 2uu^T v = v \). So \( P \) has a one-dimensional eigenspace for the eigenvalue \( \lambda = -1 \) and an \((n - 1)\)-dimensional eigenspace for the eigenvalue \( \lambda = 1 \), and thus \( \det(P) = -1 \). Also, \( P \) is an orthogonal matrix; this can be seen geometrically by noting that \( P \) is the matrix of the reflection in the hyperplane through the origin with normal vector \( u \), or by direct computation:

\[
P^T P = (I - 2(uu^T)) (1 - 2uu^T) = (1 - 2uu^T) (1 - 2uu^T)
\]

\[
= 1 - 4uu^T + 4u(u^T u)u^T = 1 - 4uu^T + 4uu^T = 1.
\]

Now recall that any orthogonal matrix has determinant \( \pm 1 \), and that the product of orthogonal matrices is orthogonal. Therefore, because \( \det(P) = -1 \), we know \( Q \) and \( PQ \) are orthogonal matrices of the same size that have opposite determinants \( \pm 1 \). The desired result now follows immediately from the following.

**Lemma.** If \( A \) is an \( n \)-by-\( n \) real orthogonal matrix such that either (i) \( \det(A) = 1 \) and \( n \) is odd or (ii) \( \det(A) = -1 \) and \( n \) is even, then 1 is an eigenvalue of \( A \).

To prove the lemma, first let \( \lambda \in \mathbb{C} \) be any eigenvalue of \( A \) and \( v \in \mathbb{C}^n \) be an associated eigenvector. Then, taking complex conjugates, \( Av = \lambda v \) yields \( A\overline{v} = \overline{\lambda} \overline{v} \), so

\[
(A\overline{v})^T A\overline{v} = \overline{\lambda} \lambda \overline{v}^T v = |\lambda|^2 |v|^2, \quad \text{while also}
\]

\[
(A\overline{v})^T A\overline{v} = \overline{v}^T (A^T A) v = \overline{v}^T v = |v|^2.
\]

Because \( |v| \neq 0 \), it follows that \( |\lambda| = 1 \). Thus the eigenvalues of \( A \) that are not 1 or \(-1 \) must occur in complex conjugate pairs for which \( \lambda \overline{\lambda} = 1 \). The product of all the eigenvalues (counting multiplicity) is \( \det(A) \), and if we leave out the complex conjugate pairs, the product of the real eigenvalues \( \pm 1 \) will still be \( \det(A) \). If \( n \) is odd, the number of real eigenvalues is odd, but to get \( \det(A) = 1 \) the number of factors \(-1 \) must be even, so the eigenvalue 1 must occur at least once. Similarly, if \( n \) is even, the number of real eigenvalues is even (in general, possibly zero), but to get \( \det(A) = -1 \) the number of factors \(-1 \) must be odd, and again the eigenvalue 1 must occur.

**B4.** Let \( F \) be the set of functions \( f(x, y) \) that are twice continuously differentiable for \( x \geq 1, y \geq 1 \) and that satisfy the following two equations (where subscripts denote partial derivatives):

\[
x f_x + y f_y = x y \ln(xy),
\]

\[
x^2 f_{xx} + y^2 f_{yy} = x y.
\]
For each \( f \in \mathcal{F} \), let
\[
m(f) = \min_{s \geq 1} \left( f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) \right).
\]

Determine \( m(f) \), and show that it is independent of the choice of \( f \in \mathcal{F} \).

**Answer.** \( m(f) = 2 \ln 2 - \frac{1}{2} \), independently of the choice of \( f \in \mathcal{F} \).

**Solution.** First note that for any \( f \in \mathcal{F} \),
\[
f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) =
= (f(s+1, s+1) - f(s, s+1)) - ((f(s+1, s) - f(s, s))
\]
\[
= \int_s^{s+1} f_x(x, s+1) dx - \int_s^{s+1} f_x(x, s) dx
\]
\[
= \int_s^{s+1} (f_x(x, s+1) - f_x(x, s)) dx
\]
\[
= \int_s^{s+1} \int_s^{s+1} f_{xy}(x, y) dy dx,
\]
so to find \( m(f) \) we must minimize this double integral. We now use the given partial differential equations to find \( f_{xy} \). Taking partial derivatives of both sides of \( xf_x + yf_y = xy \ln(xy) \) with respect to each of \( x \) and \( y \), we get the two equations
\[
f_x + xf_{xx} + yf_{yx} = y \ln(xy) + y, \quad xf_{xy} + f_y + yf_{yy} = x \ln(xy) + x. \quad (*)
\]
Note that because \( f \) is twice continuously differentiable, \( f_{yx} = f_{xy} \). If we multiply the first equation in (\( * \)) by \( x \) and the second equation by \( y \) and add the results, we obtain
\[
(xf_x + yf_y) + (x^2f_{xx} + y^2f_{yy}) + 2xyf_{xy} = 2xy \ln(xy) + 2xy.
\]
Using the two given equations to replace the bracketed expressions on the left and then dividing by \( 2xy \) leads to
\[
f_{xy} = \frac{1}{2} (\ln(xy) + 1) = \frac{1}{2} (\ln x + \ln y + 1).
\]
Therefore, we have
\[
m(f) = \min_{s \geq 1} \int_s^{s+1} \int_s^{s+1} \frac{1}{2} (\ln x + \ln y + 1) dy dx
\]
\[
= \frac{1}{2} \min_{s \geq 1} \int_s^{s+1} (\ln x + 1 + \int_s^{s+1} \ln y dy) dx
\]
\[
= \frac{1}{2} \min_{s \geq 1} \left( \int_s^{s+1} \ln x dx + 1 + \int_s^{s+1} \ln y dy \right)
\]
\[
= \min_{s \geq 1} \left( \int_s^{s+1} \ln t dt + \frac{1}{2} \right).
\]
Because the function \( \ln \) is increasing, the minimum occurs for \( s = 1 \), and so
\[
m(f) = \frac{1}{2} + \int_1^2 \ln t dt = \frac{1}{2} + (t \ln t - t)|_{t=1}^2 = 2 \ln 2 - \frac{1}{2}.
\]
Comment. With some additional calculation it can be shown that the functions in $\mathcal{F}$ are exactly those of the form

$$f(x, y) = \frac{1}{2} xy \ln(xy) - \frac{1}{2} xy + C(\ln x - \ln y) + D,$$

where $C$ and $D$ are arbitrary constants.

B5. Let $F_m$ be the $m$th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers $j$ and $k$ such that $p(2019) = F_j - F_k$.


Solution 1. More generally, let $p_N(x)$ be the polynomial of degree $N$ such that $p_N(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, N$. We will show by induction on $N$ that $p_N(2N+3) = F_{2N+3} - F_{N+2}$; setting $N = 1008$ then gives the desired answer. For the basis step, $p_1(x)$ is the linear polynomial with $p_1(1) = 1$, $p_1(3) = 2$, so $p_1(x) = (x+1)/2$ and $p_1(5) = 3 = F_5 - F_3$. To start the induction step, note that $p_N(x)$ and $p_{N-1}(x)$ have the same values for $x = 1, 3, 5, \ldots, 2N - 1$, and therefore there is a constant $c_N$ such that

$$p_N(x) = p_{N-1}(x) + c_N(x-1)(x-3) \cdots (x-(2N-1)).$$

We can find $c_N$ by substituting $x = 2N + 1$ and using the induction hypothesis $p_{N-1}(2N+1) = F_{2N+1} - F_{N+1}$, which yields

$$F_{2N+1} = F_{2N+1} - F_{N+1} + c_N(2N)(2N-2) \cdots 2 \quad \text{and thus} \quad c_N = \frac{F_{N+1}}{2N N!}.$$ 

It follows that

$$p_N(x) = (x+1)/2 + c_2(x-1)(x-3) + \cdots + c_N(x-1)(x-3) \cdots (x-(2N-1))$$

$$= (x+1)/2 + \sum_{i=2}^{N} \frac{F_{i+1}}{2i!} (x-1)(x-3) \cdots (x-(2i-1)),$$

and in particular

$$p_N(2N+3) = N + 2 + \sum_{i=2}^{N} \frac{F_{i+1}(2N+2)(2N) \cdots (2N-2i+4)}{2i!}$$

$$= N + 2 + \sum_{i=2}^{N} \frac{F_{i+1}(N+1)N \cdots (N-i+2)}{i!}$$

$$= N + 2 + \sum_{i=2}^{N} F_{i+1} \binom{N+1}{i}$$

$$= \sum_{i=0}^{N} F_{i+1} \binom{N+1}{i} = \sum_{i=0}^{N+1} F_{i+1} \binom{N+1}{i} - F_{N+2}.$$
Thus the induction on $N$ will be complete if we can prove that $\sum_{i=0}^{K} F_{i+1} \binom{K}{i} = F_{2K+1}$ for any positive integer $K$. This in turn follows from the more general fact

$$\sum_{i=0}^{K} F_{i+m} \binom{K}{i} = F_{2K+m},$$

which is true for all positive integers $K$ and $m$ and can be shown by a relatively straightforward induction on $K$ (the generality helps because the induction step uses the induction hypothesis both for $m$ and for $m+1$).

**Solution 2.** By Binet’s formula, we have

$$F_{2n+1} = \frac{1}{\sqrt{5}} \left( \left( 1 + \sqrt{5} \right)^{2n+1} - \left( 1 - \sqrt{5} \right)^{2n+1} \right)$$

$$= \frac{1 + \sqrt{5}}{2\sqrt{5}} r^n_1 - \frac{1 - \sqrt{5}}{2\sqrt{5}} r^n_2,$$

where $r_1, r_2$ are given by

$$r_1 = \frac{3 + \sqrt{5}}{2} = \left( \frac{1 + \sqrt{5}}{2} \right)^2, \quad r_2 = \frac{3 - \sqrt{5}}{2} = \left( \frac{1 - \sqrt{5}}{2} \right)^2.$$

Therefore, if we define $q_1(x), q_2(x)$ to be the polynomials of degree 1008 such that

$$q_1(2n+1) = r^n_1 \quad \text{and} \quad q_2(2n+1) = r^n_2 \quad \text{for} \quad n = 0, 1, 2, \ldots, 1008,$$

we will have

$$p(x) = \frac{1 + \sqrt{5}}{2\sqrt{5}} q_1(x) - \frac{1 - \sqrt{5}}{2\sqrt{5}} q_2(x).$$

Thus the following fact about interpolating polynomials will be useful.

**Lemma.** If $q(x)$ is the polynomial of degree $N$ such that $q(n) = r^n$ for $n = 0, 1, 2, \ldots N$, where $r$ is some fixed real number, then $q(N+1) = r^{N+1} - (r-1)^{N+1}$.

To prove the lemma, first define

$$T_{N,k}(x) = \prod_{j=0}^{N} (x - j) \quad \text{and} \quad Q_{N,k}(x) = \frac{T_{N,k}(x)}{T_{N,k}(k)}.$$

Then $Q_{N,k}(x)$ is the polynomial of degree $N$ such that for integers $n$ with $0 \leq n \leq N$, we have $Q_{N,k}(n) = \delta_{n,k}$, where $\delta$ is the Kronecker delta. Therefore, $q(x)$ is the linear combination

$$q(x) = \sum_{k=0}^{N} r^k Q_{N,k}(x).$$
of these “basic” interpolating polynomials. We then get
\[ q(N + 1) = \sum_{k=0}^{N} r_k Q_{N,k}(N + 1) = \sum_{k=0}^{N} r_k \frac{T_{N,k}(N + 1)}{T_{N,k}(k)} \]
\[ = \sum_{k=0}^{N} r_k \frac{(N + 1)!/(N + 1 - k)}{k!(-1)^{N-k} (N-k)!} \]
\[ = \sum_{k=0}^{N} (-1)^{N-k} r_k \frac{(N + 1)!}{k! (N + 1 - k)!} = \sum_{k=0}^{N} (-1)^{N-k} r_k \binom{N+1}{k} \]
\[ = -\sum_{k=0}^{N+1} (-1)^{N+1-k} r_k \binom{N+1}{k} + r^{N+1} = r^{N+1} - (r - 1)^{N+1}, \]
proving the lemma.

The lemma applies to the polynomials \( q(x) = q_1(2x + 1) \) and \( q(x) = q_2(2x + 1) \), so we can compute

\[ p(2019) = \frac{1 + \sqrt{5}}{2\sqrt{5}} q_1(2019) - \frac{1 - \sqrt{5}}{2\sqrt{5}} q_2(2019) \]
\[ = \frac{1 + \sqrt{5}}{2\sqrt{5}} (r_1^{1009} - (r_1 - 1)^{1009}) - \frac{1 - \sqrt{5}}{2\sqrt{5}} (r_2^{1009} - (r_2 - 1)^{1009}) \]
\[ = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{2019} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2019} \right) - \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{1010} - \left( \frac{1 - \sqrt{5}}{2} \right)^{1010} \right) \]
\[ = F_{2019} - F_{1010}, \]
where we have used that \( r_1 - 1 = \frac{1 + \sqrt{5}}{2} \) and \( r_2 - 1 = \frac{1 - \sqrt{5}}{2} \).

B6. Let \( \mathbb{Z}^n \) be the integer lattice in \( \mathbb{R}^n \). Two points in \( \mathbb{Z}^n \) are called neighbors if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers \( n \geq 1 \) does there exist a set of points \( S \subset \mathbb{Z}^n \) satisfying the following two conditions?

(1) If \( p \) is in \( S \), then none of the neighbors of \( p \) is in \( S \).
(2) If \( p \in \mathbb{Z}^n \) is not in \( S \), then exactly one of the neighbors of \( p \) is in \( S \).

**Solution.** We will show how to construct such a subset for every \( n \). Because each point in \( \mathbb{Z}^n \) has exactly 2\( n \) neighbors, for each point there is a set of size 2\( n + 1 \) (consisting of its neighbors and itself) of which exactly one element should be in \( S \). This may suggest looking at congruences modulo 2\( n + 1 \). More specifically, for each integer \( k \) with 0 \leq k \leq 2n we can define a subset \( S_k \) of \( \mathbb{Z}^n \) by

\[ S_k = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n | x_1 + 3x_2 + 5x_3 + \cdots + (2n - 1)x_n \equiv k \mod (2n + 1)\}. \]

It is immediate that these 2\( n + 1 \) subsets partition \( \mathbb{Z}^n \); we claim that any of the subsets has the desired properties for \( S \). To see this, let

\[ f(x_1, x_2, \ldots, x_n) = x_1 + 3x_2 + \cdots + (2n - 1)x_n, \quad \text{so that} \]
\[ S_k = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) \equiv k \mod (2n + 1)\}. \]
Note that moving from a point \( p = (x_1, \ldots, x_n) \) to one of its neighbors adds one of the numbers \( \pm 1, \pm 3, \ldots, \pm (2n - 1) \) to the value of \( f(x_1, \ldots, x_n) \). Because these numbers represent all the nonzero
congruence classes mod $(2n + 1)$
(specifically, $1 \equiv 1$, $2 \equiv -(2n-1)$, $3 \equiv 3$, $4 \equiv -(2n-3)$, \ldots, $2n-1 \equiv 2n-1$, $2n \equiv -1$),
for any $k$ exactly one of the point $p$ and its $2n$ neighbors is guaranteed to be in the set $S_k$, as desired.