B1. Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Answer. $5n + 1$.

Solution. Let S_k be the set of all points $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 = 2^k$, so that

$$P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{n} S_k\,.$$

Then $S_0 = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$ and $S_1 = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. For $k \geq 2$ and any $(x, y) \in S_k$, we have $x^2 + y^2 \equiv 0 \pmod 4$, so because 0 and 1 are the only squares (mod 4), x and y must both be even. If we put $x = 2x_1, y = 2y_1$, then $4(x_1^2 + y_1^2) = 2^k$, so $x_1^2 + y_1^2 = 2^{k-2}$ and $(x_1, y_1) \in S_{k-2}$. It follows by induction on k that

$$S_k = \{(2^q, 0), (-2^q, 0), (0, 2^q), (0, -2^q)\} \quad \text{for } k = 2q \text{ even} \quad \text{and}$$
$$S_k = \{(2^t, 2^t), (2^t, -2^t), (-2^t, 2^t), (-2^t, -2^t)\} \quad \text{for } k = 2t + 1 \text{ odd.}$$

Note that for any k, the four points of S_k form the vertices of a square; also, for any q there are four squares with one vertex at the origin, two vertices in S_{2q}, and one vertex in S_{2q+1} (the square with vertices $(0, 0), (2^q, 0), (2^q, 2^q), (0, 2^q)$ and the three squares obtained from it by rotation through $\pi/2, \pi, 3\pi/2$ around the origin), and for any t there are four squares with one vertex at the origin, two vertices in S_{2t+1}, and one vertex in S_{2t+2} (the square with vertices $(0, 0), (2^t, 2^t), (0, 2^{t+1}), (-2^t, 2^t)$ and the three squares obtained from it by rotation). Thus when we pass from P_n to P_{n+1} by including the points in S_n, we get at least five additional squares, whether n is even or odd. Because there is exactly one four-point subset of P_0 (namely S_0) that gives a square, there will be exactly $5n + 1$ such subsets of P_n , provided that the only squares of which all vertices are in the set

$$P_\infty = \bigcup_{n=0}^{\infty} P_n = \{(0, 0)\} \cup \bigcup_{k=0}^{\infty} S_k$$

are the ones we have mentioned so far.

To see that there are no additional such squares, first note that for all $k \geq 2$, all points in S_k have only even coordinates; if we have a square for which each vertex is in $\{(0, 0)\} \cup \bigcup_{k=2}^{\infty} S_k$, we can scale down all coordinates by a factor 2 and get another square of which all vertices are in P_∞ . Thus it is sufficient to consider squares for which all vertices are in P_∞ and at least one vertex is in $S_0 \cup S_1$.

It is impossible to have just *one* of the vertices of such a square be in $S_0 \cup S_1$, because the square of the side length from that vertex to any other vertex would be 1 or 2 mod 4, whereas the square of a side length not involving that vertex would be 0 mod 4. By the same argument, if exactly *two* of the vertices of such a square are in $S_0 \cup S_1$, those two must be opposite vertices of the square. And if three or four of the vertices of such a square are in $S_0 \cup S_1$, we can choose two such vertices that are opposite each other. Thus it is enough to analyze squares of which all vertices are in P_∞ and two opposite vertices are in $S_0 \cup S_1$.

If one of the two opposite vertices in $S_0 \cup S_1$ is in S_0, up to rotational symmetry we can assume it is $(1, 0)$. Then it can be checked by a quick case analysis that the vertex of the square opposite it cannot be in S_1; if it is $(-1, 0)$, then the vertices of the square are the four points of S_0, otherwise it is $(0, 1)$ up to reflectional symmetry, and the vertices of the square are $(0, 0), (1, 0), (1, 1), (0, 1)$. The final possibility is that the two opposite vertices in $S_0 \cup S_1$ are both in S_1, in which case we can assume up to symmetry that they are $(1, 1)$ and $(-1, -1)$ (and the vertices of the square are all the points of S_1) or $(1, 1)$ and $(-1, 1)$ (and the vertices of the square are $(0, 0), (1, 1), (0, 2), (-1, 1)$). We have now checked that the only possible squares whose vertices are all in P_n are the $5n + 1$ squares found above.

B2. For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right) \cos^2\left(\frac{k\pi}{2n}\right)} .$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3} .$$

Answer. $\dfrac{8}{\pi^3}$.

Solution. Let $\theta_n = \dfrac{\pi}{2n}$, and note that $\sin\theta_n \neq 0$. Then we have

$$\begin{aligned}
a_n &= \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{\sin((2k-1)\theta_n)\sin\theta_n}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{1}{2} \frac{\cos((2k-2)\theta_n) - \cos(2k\theta_n)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \frac{1}{2} \frac{2\cos^2((k-1)\theta_n) - 1 - (2\cos^2(k\theta_n) - 1)}{\cos^2((k-1)\theta_n)\ \cos^2(k\theta_n)} \\
&= \frac{1}{\sin\theta_n} \sum_{k=1}^{n-1} \left(\frac{1}{\cos^2(k\theta_n)} - \frac{1}{\cos^2((k-1)\theta_n)}\right).
\end{aligned}$$

We now see that the sum telescopes, and we get

$$a_n = \frac{1}{\sin\theta_n}\left(\frac{1}{\cos^2((n-1)\theta_n)} - 1\right).$$

Because $n\theta_n = \dfrac{\pi}{2}$, we have $\cos((n-1)\theta_n) = \cos(\dfrac{\pi}{2} - \theta_n) = \sin\theta_n$, so

$$a_n = \frac{1}{\sin^3\theta_n} - \frac{1}{\sin\theta_n}.$$

Now let $n \to \infty$. Then $\theta_n \to 0$, so

$$\lim_{n\to\infty} n \sin\theta_n = \lim_{n\to\infty} n\theta_n = \frac{\pi}{2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{n^3} = \lim_{n\to\infty} \frac{1}{(n\sin\theta_n)^3} - \lim_{n\to\infty} \frac{1}{n^2(n\sin\theta_n)}$$
$$= \frac{1}{(\pi/2)^3} - 0 = \frac{8}{\pi^3}.$$

B3. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

Solution. Note that $P(u) = u - 2uu^Tu = u - 2u = -u$, while if $v \in \mathbb{R}^n$ is a vector orthogonal to u, that is, if $u^Tv = 0$, we have $P(v) = v - 2uu^Tv = v$. So P has a one-dimensional eigenspace for the eigenvalue $\lambda = -1$ and an $(n-1)$-dimensional eigenspace for the eigenvalue $\lambda = 1$, and thus $\det(P) = -1$. Also, P is an orthogonal matrix; this can be seen geometrically by noting that P is the matrix of the reflection in the hyperplane through the origin with normal vector u, or by direct computation:

$$P^TP = (I - 2(uu^T)^T)(1 - 2uu^T) = (1 - 2uu^T)(1 - 2uu^T)$$
$$= 1 - 4uu^T + 4u(u^Tu)u^T = 1 - 4uu^T + 4uu^T = 1.$$

Now recall that any orthogonal matrix has determinant ± 1, and that the product of orthogonal matrices is orthogonal. Therefore, because $\det(P) = -1$, we know Q and PQ are orthogonal matrices of the same size that have opposite determinants ± 1. The desired result now follows immediately from the following.

Lemma. If A is an n-by-n real orthogonal matrix such that either (i) $\det(A) = 1$ and n is odd or (ii) $\det(A) = -1$ and n is even, then 1 is an eigenvalue of A.

To prove the lemma, first let $\lambda \in \mathbb{C}$ be any eigenvalue of A and $v \in \mathbb{C}^n$ be an associated eigenvector. Then, taking complex conjugates, $Av = \lambda v$ yields $A\overline{v} = \overline{\lambda}\overline{v}$, so

$$(A\overline{v})^T Av = \overline{\lambda}\lambda\, \overline{v}^T v = |\lambda|^2 |v|^2, \quad \text{while also}$$
$$(A\overline{v})^T Av = \overline{v}^T(A^TA)v = \overline{v}^T v = |v|^2.$$

Because $|v| \neq 0$, it follows that $|\lambda| = 1$. Thus the eigenvalues of A that are not 1 or -1 must occur in complex conjugate pairs for which $\lambda\overline{\lambda} = 1$. The product of all the eigenvalues (counting multiplicity) is $\det(A)$, and if we leave out the complex conjugate pairs, the product of the real eigenvalues ± 1 will still be $\det(A)$. If n is odd, the number of real eigenvalues is odd, but to get $\det(A) = 1$ the number of factors -1 must be even, so the eigenvalue 1 must occur at least once. Similarly, if n is even, the number of real eigenvalues is even (in general, possibly zero), but to get $\det(A) = -1$ the number of factors -1 must be odd, and again the eigenvalue 1 must occur.

B4. Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$xf_x + yf_y = xy\ln(xy),$$
$$x^2 f_{xx} + y^2 f_{yy} = xy.$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s \geq 1} \Big(f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) \Big).$$

Determine $m(f)$, and show that it is independent of the choice of f.

Answer. $m(f) = 2 \ln 2 - \dfrac{1}{2}$, independently of the choice of $f \in \mathcal{F}$.

Solution. First note that for any $f \in \mathcal{F}$,

$$\begin{aligned} f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) &= \\ &= (f(s+1, s+1) - f(s, s+1)) - ((f(s+1, s) - f(s, s)) \\ &= \int_s^{s+1} f_x(x, s+1)\, dx - \int_s^{s+1} f_x(x, s)\, dx \\ &= \int_s^{s+1} (f_x(x, s+1) - f_x(x, s))\, dx \\ &= \int_s^{s+1} \int_s^{s+1} f_{xy}(x, y)\, dy\, dx, \end{aligned}$$

so to find $m(f)$ we must minimize this double integral. We now use the given partial differential equations to find f_{xy}. Taking partial derivatives of both sides of $x f_x + y f_y = xy \ln(xy)$ with respect to each of x and y, we get the two equations

$$f_x + x f_{xx} + y f_{yx} = y \ln(xy) + y, \quad x f_{xy} + f_y + y f_{yy} = x \ln(xy) + x. \quad (*)$$

Note that because f is twice continuously differentiable, $f_{yx} = f_{xy}$. If we multiply the first equation in $(*)$ by x and the second equation by y and add the results, we obtain

$$(x f_x + y f_y) + (x^2 f_{xx} + y^2 f_{yy}) + 2xy f_{xy} = 2xy \ln(xy) + 2xy.$$

Using the two given equations to replace the bracketed expressions on the left and then dividing by $2xy$ leads to

$$f_{xy} = \frac{1}{2}(\ln(xy) + 1) = \frac{1}{2}(\ln x + \ln y + 1).$$

Therefore, we have

$$\begin{aligned} m(f) &= \min_{s \geq 1} \int_s^{s+1} \int_s^{s+1} \frac{1}{2}(\ln x + \ln y + 1)\, dy\, dx \\ &= \frac{1}{2} \min_{s \geq 1} \int_s^{s+1} \Big(\ln x + 1 + \int_s^{s+1} \ln y\, dy \Big)\, dx \\ &= \frac{1}{2} \min_{s \geq 1} \Big(\int_s^{s+1} \ln x\, dx + 1 + \int_s^{s+1} \ln y\, dy \Big) \\ &= \min_{s \geq 1} \Big(\int_s^{s+1} \ln t\, dt + \frac{1}{2} \Big). \end{aligned}$$

Because the function ln is increasing, the minimum occurs for $s = 1$, and so

$$m(f) = \frac{1}{2} + \int_1^2 \ln t\, dt = \frac{1}{2} + (t \ln t - t)\Big|_{t=1}^{2} = 2 \ln 2 - \frac{1}{2}.$$

Comment. With some additional calculation it can be shown that the functions in $\mathcal{F}$ are exactly those of the form

$$f(x,y) = \frac{1}{2}xy\ln(xy) - \frac{1}{2}xy + C(\ln x - \ln y) + D,$$

where C and D are arbitrary constants.

B5. Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

Answer. $p(2019) = F_{2019} - F_{1010}$, so $j = 2019, k = 1010$.

Solution 1. More generally, let $p_N(x)$ be the polynomial of degree N such that $p_N(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, N$. We will show by induction on N that $p_N(2N+3) = F_{2N+3} - F_{N+2}$; setting $N = 1008$ then gives the desired answer. For the basis step, $p_1(x)$ is the linear polynomial with $p_1(1) = 1$, $p_1(3) = 2$, so $p_1(x) = (x+1)/2$ and $p_1(5) = 3 = F_5 - F_3$. To start the induction step, note that $p_N(x)$ and $p_{N-1}(x)$ have the same values for $x = 1, 3, 5, \ldots, 2N-1$, and therefore there is a constant c_N such that

$$p_N(x) = p_{N-1}(x) + c_N(x-1)(x-3)\cdots(x-(2N-1)).$$

We can find c_N by substituting $x = 2N+1$ and using the induction hypothesis $p_{N-1}(2N+1) = F_{2N+1} - F_{N+1}$, which yields

$$F_{2N+1} = F_{2N+1} - F_{N+1} + c_N(2N)(2N-2)\cdots 2 \quad \text{and thus} \quad c_N = \frac{F_{N+1}}{2^N N!}.$$

It follows that

$$\begin{aligned} p_N(x) &= (x+1)/2 + c_2(x-1)(x-3) + \cdots + c_N(x-1)(x-3)\cdots(x-(2N-1)) \\ &= (x+1)/2 + \sum_{i=2}^{N} \frac{F_{i+1}}{2^i i!}(x-1)(x-3)\cdots(x-(2i-1)), \end{aligned}$$

and in particular

$$\begin{aligned} p_N(2N+3) &= N + 2 + \sum_{i=2}^{N} \frac{F_{i+1}}{2^i i!}(2N+2)(2N)\cdots(2N-2i+4) \\ &= N + 2 + \sum_{i=2}^{N} \frac{F_{i+1}(N+1)N\cdots(N-i+2)}{i!} \\ &= N + 2 + \sum_{i=2}^{N} F_{i+1}\binom{N+1}{i} \\ &= \sum_{i=0}^{N} F_{i+1}\binom{N+1}{i} = \sum_{i=0}^{N+1} F_{i+1}\binom{N+1}{i} - F_{N+2}. \end{aligned}$$

Thus the induction on N will be complete if we can prove that $\sum_{i=0}^{K} F_{i+1}\binom{K}{i} = F_{2K+1}$ for any positive integer K. This in turn follows from the more general fact

$$\sum_{i=0}^{K} F_{i+m}\binom{K}{i} = F_{2K+m}\,,$$

which is true for all positive integers K and m and can be shown by a relatively straightforward induction on K (the generality helps because the induction step uses the induction hypothesis both for m and for $m+1$).

Solution 2. By Binet's formula, we have

$$\begin{aligned} F_{2n+1} &= \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^{2n+1} - \big(\frac{1-\sqrt{5}}{2}\big)^{2n+1}\Big) \\ &= \frac{1+\sqrt{5}}{2\sqrt{5}}\,r_1^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\,r_2^n\,, \end{aligned}$$

where r_1, r_2 are given by

$$r_1 = \frac{3+\sqrt{5}}{2} = \big(\frac{1+\sqrt{5}}{2}\big)^2, \quad r_2 = \frac{3-\sqrt{5}}{2} = \big(\frac{1-\sqrt{5}}{2}\big)^2.$$

Therefore, if we define $q_1(x), q_2(x)$ to be the polynomials of degree 1008 such that

$$q_1(2n+1) = r_1^n \quad \text{and} \quad q_2(2n+1) = r_2^n \quad \text{for} \quad n = 0, 1, 2, \ldots, 1008,$$

we will have

$$p(x) = \frac{1+\sqrt{5}}{2\sqrt{5}}\,q_1(x) \; - \; \frac{1-\sqrt{5}}{2\sqrt{5}}\,q_2(x).$$

Thus the following fact about interpolating polynomials will be useful.

Lemma. If $q(x)$ is the polynomial of degree N such that $q(n) = r^n$ for $n = 0, 1, 2, \ldots N$, where r is some fixed real number, then $q(N+1) = r^{N+1} - (r-1)^{N+1}$.

To prove the lemma, first define

$$T_{N,k}(x) = \prod_{\substack{j=0 \\ j\neq k}}^{N} (x-j) \quad \text{and} \quad Q_{N,k}(x) = \frac{T_{N,k}(x)}{T_{N,k}(k)}\,.$$

Then $Q_{N,k}(x)$ is the polynomial of degree N such that for integers n with $0 \le n \le N$, we have $Q_{N,k}(n) = \delta_{n,k}$, where δ is the Kronecker delta. Therefore, $q(x)$ is the linear combination

$$q(x) = \sum_{k=0}^{N} r^k Q_{N,k}(x)$$

of these "basic" interpolating polynomials. We then get

$$\begin{aligned}
q(N+1) &= \sum_{k=0}^{N} r^k\, Q_{N,k}(N+1) = \sum_{k=0}^{N} r^k\, \frac{T_{N,k}(N+1)}{T_{N,k}(k)} \\
&= \sum_{k=0}^{N} r^k\, \frac{(N+1)!/(N+1-k)}{k!\,(-1)^{N-k}\,(N-k)!} \\
&= \sum_{k=0}^{N} (-1)^{N-k} r^k\, \frac{(N+1)!}{k!\,(N+1-k)!} = \sum_{k=0}^{N} (-1)^{N-k} r^k \binom{N+1}{k} \\
&= -\sum_{k=0}^{N+1} (-1)^{N+1-k} r^k \binom{N+1}{k} + r^{N+1} = r^{N+1} - (r-1)^{N+1},
\end{aligned}$$

proving the lemma.

The lemma applies to the polynomials $q(x) = q_1(2x+1)$ and $q(x) = q_2(2x+1)$, so we can compute

$$\begin{aligned}
p(2019) &= \frac{1+\sqrt{5}}{2\sqrt{5}}\, q_1(2019) \;-\; \frac{1-\sqrt{5}}{2\sqrt{5}}\, q_2(2019) \\
&= \frac{1+\sqrt{5}}{2\sqrt{5}} \big(r_1^{1009} - (r_1-1)^{1009}\big) \;-\; \frac{1-\sqrt{5}}{2\sqrt{5}} \big(r_2^{1009} - (r_2-1)^{1009}\big) \\
&= \frac{1}{\sqrt{5}} \Big(\big(\tfrac{1+\sqrt{5}}{2}\big)^{2019} - \big(\tfrac{1-\sqrt{5}}{2}\big)^{2019} \Big) - \frac{1}{\sqrt{5}} \Big(\big(\tfrac{1+\sqrt{5}}{2}\big)^{1010} - \big(\tfrac{1-\sqrt{5}}{2}\big)^{1010} \Big) \\
&= F_{2019} - F_{1010}\,,
\end{aligned}$$

where we have used that $r_1 - 1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 - 1 = \dfrac{1-\sqrt{5}}{2}$.

B6. Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \ge 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If p is in S, then none of the neighbors of p is in S.

(2) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.

Solution. We will show how to construct such a subset for every n. Because each point in $\mathbb{Z}^n$ has exactly $2n$ neighbors, for each point there is a set of size $2n+1$ (consisting of its neighbors and itself) of which exactly one element should be in S. This may suggest looking at congruences modulo $2n+1$. More specifically, for each integer k with $0 \le k \le 2n$ we can define a subset S_k of $\mathbb{Z}^n$ by

$$S_k = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + 3x_2 + 5x_3 + \cdots + (2n-1)x_n \equiv k \mod (2n+1)\}.$$

It is immediate that these $2n+1$ subsets partition $\mathbb{Z}^n$; we claim that any of the subsets has the desired properties for S. To see this, let

$f(x_1, x_2, \ldots, x_n) = x_1 + 3x_2 + \cdots + (2n-1)x_n$, so that

$S_k = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) \equiv k \mod (2n+1)\}$. Note that moving from a point $p = (x_1, \ldots, x_n)$ to one of its neighbors adds one of the numbers $\pm 1, \pm 3, \ldots, \pm(2n-1)$ to the value of $f(x_1, \ldots, x_n)$. Because these numbers represent all the nonzero

congruence classes mod $(2n+1)$
(specifically, $1 \equiv 1, 2 \equiv -(2n-1), 3 \equiv 3, 4 \equiv -(2n-3), \ldots, 2n-1 \equiv 2n-1, 2n \equiv -1$), for any k exactly one of the point p and its $2n$ neighbors is guaranteed to be in the set S_k, as desired.