A1. How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.
(ii) N has at most 2020 decimal digits.
(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Answer. $504 \cdot 1009 = 508536$.

Solution 1. A positive integer N satisfying (iii), with j ones followed by k zeros, has the form

$$N = \frac{10^j - 1}{9} \cdot 10^k$$

where $j \geq 1$, $k \geq 0$, and $j + k \leq 2020$. Note that $2020 = 20 \cdot 101$, so to satisfy (i) the integer N must be divisible by 101 and end in at least two zeros (so $k \geq 2$). If 101 divides N then 101 divides $M = 10^j - 1$. A quick check shows that $M \equiv 0, 9, 99, 90 \mod 101$ when $j \equiv 0, 1, 2, 3 \mod 4$. Consequently, 4 must divide j. (One can see directly that the conditions $k \geq 2, 4|j$ are necessary and sufficient by noting that 101 divides 1111 but not 1, 11 or 111.)

If $j = 4m$, then for N to satisfy (ii) also, we need $2 \leq k \leq 2020 - 4m$, for a total of $2019 - 4m$ possible values of k. The total number of integers N satisfying all the conditions is therefore

$$\sum_{m=1}^{504}(2019 - 4m) = 2019 \cdot 504 - 4 \cdot \frac{504 \cdot 505}{2} = 504 \cdot (2019 - 1010) = 504 \cdot 1009 = 508536.$$

Solution 2. As in the first solution, it is straightforward to show that the acceptable numbers N are those for which there are at most 2020 decimal digits, consisting of j ones with $4|j$ followed by k zeros with $k \geq 2$. By introducing additional "phantom" digits z at the beginning of the number, we can convert it to a string of length exactly 2020 of the form $zzz \cdots z111 \cdots 1000 \cdots 0$. We now show that the set of such strings is in bijective correspondence with a set of size $\binom{1009}{2} = 508536$. To see this, remove the final two zeros from the string, and group the remaining 2018 positions in the string into consecutive pairs. Then any choice of 2 of these 1009 pairs corresponds to a unique string of the desired form, as follows. If the two chosen pairs have an even number of pairs between them, put a z in each position before the first chosen pair, put 11 for each of the chosen pairs and all pairs in between, and put a 0 in each position after the second chosen pair, for example:

$$\underbrace{xx}\ \underbrace{xx}_{\text{Choose}}\ \underbrace{xx}\ \underbrace{xx}\ \underbrace{xx}_{\text{Choose}}\ \underbrace{xx} \cdots \mapsto zz \mid 11 \mid 11 \mid 11 \mid 11 \mid 00 \ \cdots .$$

If the two chosen pairs are separated by an odd number of pairs, do the same except for replacing the chosen pairs by $z1$ and 10, respectively, for example:

$$\underbrace{xx}\ \underbrace{xx}_{\text{Choose}}\ \underbrace{xx}\ \underbrace{xx}_{\text{Choose}}\ \underbrace{xx} \cdots \mapsto zz \mid z1 \mid 11 \mid 10 \mid 00 \ \cdots .$$

Note that in either case, the resulting number of ones is divisible by 4. Erasing the digits z and restoring the two zeros that were removed at the end of the string, we get every acceptable number N exactly once from some choice of 2 of the 1009 consecutive pairs.

Solution 3. As in the first solution, the positive integers N satisfying conditions (i) and (iii) have j ones followed by k zeros, with $4|j$, $j \geq 1$, and $k \geq 2$. Thus if we let b_m be the number of m-digit positive integers with these properties, we have the generating function

$$\sum_{m=0}^{\infty} b_m x^m = \frac{x^4}{1-x^4} \cdot \frac{x^2}{1-x} .$$

Hence the generating function for the number $B_m = \sum_{k \leq m} b_k$ of such integers with at most m digits is

$$\sum_{m=0}^{\infty} B_m x^m = \frac{1}{1-x} \cdot \sum_{m=0}^{\infty} b_m x^m = \frac{x^6}{(1-x)^2(1-x^4)} = \frac{x^6(1+x+x^2+x^3)^2}{(1-x^4)^3}.$$

Because

$$\frac{1}{(1-y)^3} = \sum_{k=0}^{\infty} \binom{k+2}{2} y^k$$

and $x^6(1+x+x^2+x^3)^2 = x^6 + 2x^7 + 3x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$, we can read off the answer

$$B_{2020} = \binom{504}{2} + 3\binom{505}{2} = 126756 + 381780 = 508536.$$

A2. Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

Answer. $2^{2k} = 4^k$.

Solution 1. Consider lattice paths of length $2k+1$ that begin from the origin, and where each step is either of the form $(x, y) \mapsto (x+1, y)$ or $(x, y) \mapsto (x, y+1)$; thus these paths will be in the first quadrant. There are 2^{2k+1} such paths, and by reflective symmetry, exactly half of them reach the line $x = k+1$ (the other half reach the line $y = k+1$). So there are 2^{2k} such paths that have at least $k+1$ "right steps" among their $2k+1$ steps.

On the other hand, any of those paths first touches the line $x = k+1$ at some point $(k+1, j)$, which means that the previous step came from (k, j). There are exactly $\binom{k+j}{j}$ paths from $(0, 0)$ to (k, j), and 2^{k-j} possible paths after $(k+1, j)$. Summing over the possible values for j, which range from 0 to k, gives the sum from the problem statement, and thus that sum equals 2^{2k}.

Solution 2. Let $S(k)$ denote the sum from the problem statement. Then using basic properties of binomial coefficients, one finds that for $k \geq 0$,

$$\begin{aligned} S(k+1) &= \sum_{j=0}^{k+1} 2^{k+1-j} \binom{k+1+j}{j} \\ &= \sum_{j=0}^{k+1} 2^{k+1-j} \left(\binom{k+j}{j} + \binom{k+j}{j-1} \right) \\ &= 2 \sum_{j=0}^{k+1} 2^{k-j} \binom{k+j}{j} + \sum_{j=0}^{k} 2^{k-j} \binom{k+j+1}{j} \\ &= 2\, S(k) + \binom{2k+1}{k+1} + \frac{1}{2} \left(S(k+1) - \binom{2k+2}{k+1} \right) \\ &= 2\, S(k) + \frac{1}{2}\, S(k+1) + \binom{2k+1}{k+1} - \frac{1}{2} \binom{2k+2}{k+1} \\ &= 2\, S(k) + \frac{1}{2}\, S(k+1). \end{aligned}$$

Therefore $S(k+1) = 4\, S(k)$, and since $S(0) = 1$, by induction we have $S(k) = 4^k$ for all k.

Solution 3. Note that the desired sum

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j} = 2^k \sum_{j=0}^{k} 2^{-j} \binom{k+j}{k}$$

is the coefficient of x^k in the polynomial

$$\begin{aligned} P_k(x) &= 2^k \sum_{j=0}^{k} 2^{-j} (1+x)^{k+j} \\ &= 2^k (1+x)^k \sum_{j=0}^{k} \left(\frac{1+x}{2} \right)^j \\ &= 2^k (1+x)^k \, \frac{1 - \left(\frac{1+x}{2}\right)^{k+1}}{1 - \frac{1+x}{2}} \\ &= 2^{k+1} (1+x)^k \, \frac{1 - \left(\frac{1+x}{2}\right)^{k+1}}{1-x} \\ &= \left[2^{k+1}(1+x)^k - (1+x)^{2k+1} \right] \frac{1}{1-x} \\ &= \left[2^{k+1}(1+x)^k - (1+x)^{2k+1} \right] (1 + x + x^2 + \ldots). \end{aligned}$$

But this coefficient can also be expressed as

$$2^{k+1} \sum_{j=0}^{k} \binom{k}{j} - \sum_{j=0}^{k} \binom{2k+1}{j} = 2^{k+1} \cdot 2^k - \frac{1}{2} \cdot 2^{2k+1} = 2^{2k} = 4^k\,,$$

as claimed.

A3. Let $a_0 = \pi/2$, and for $n \geq 1$, let $a_n = \sin(a_{n-1})$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Answer. The series diverges.

Solution 1. Note that $a_1 = 1$; we now show by induction on n that for all $n \geq 1$, $a_n \geq 1/\sqrt{n}$. Note that on the interval $(0, \pi/2)$, $\sin(x)$ is increasing and $\sin x > x - x^3/6$ by Taylor's theorem with remainder (because the fifth derivative, $\cos x$, is positive on the interval). In particular, from the induction hypothesis,

$$a_{n+1} = \sin a_n \geq \sin\left(\frac{1}{\sqrt{n}}\right) > \frac{1}{\sqrt{n}} - \frac{1}{6}\left(\frac{1}{\sqrt{n}}\right)^3.$$

On the other hand,

$$\begin{aligned}\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} \\ &= \frac{1}{(\sqrt{n+1}+\sqrt{n})\sqrt{n}\sqrt{n+1}} \\ &> \frac{1}{(3\sqrt{n})\sqrt{n}(2\sqrt{n})} = \frac{1}{6}\left(\frac{1}{\sqrt{n}}\right)^3,\end{aligned}$$

so

$$a_{n+1} > \frac{1}{\sqrt{n+1}},$$

completing the induction. But then $\sum_{n=1}^{\infty} a_n^2$ diverges because it is greater than the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Solution 2. Because we have $0 < \sin x < x$ for $x \in (0, \pi/2]$, the sequence (a_n) is monotonically decreasing to a limit $L \in [0, 1]$. By the continuity of the sine function we must have $L = \sin(L)$, so $L = 0$. Now it follows from L'Hôpital's rule that

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right) &= \lim_{n\to\infty}\frac{a_n^2 - \sin^2(a_n)}{a_n^2\sin^2(a_n)} = \lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x} \\ &= \lim_{x\to 0}\frac{x-\sin x}{x^3}\cdot\frac{x+\sin x}{x}\cdot\left(\frac{x}{\sin x}\right)^2 = \frac{1}{6}\cdot 2\cdot 1^2 = \frac{1}{3}.\end{aligned}$$

We can then apply the Stolz-Cesàro theorem to get

$$\lim_{n\to\infty}\frac{1/a_n^2}{n} = \lim_{n\to\infty}\frac{1/a_{n+1}^2 - 1/a_n^2}{(n+1)-n} = \frac{1}{3}.$$

Hence $a_n^2 \sim 3/n$ as $n \to \infty$ and the given series diverges.

A4. Consider a horizontal strip of $N + 2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color this neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining.

Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

Answer. $\dfrac{1}{e}$.

Solution. Note that $w(0) = 0$, $w(1) = 1$, and $w(2) = 1$ (as eventually one of the two white squares will turn black). Let $N \geq 3$, and number the original white squares from 1 to N. For the first step of the process, there are $2N$ equally likely possible "outcomes", consisting of a choice of a white square along with a choice of one of its neighbors to be colored. Two of these outcomes (when white square 1 or N is chosen along with its neighbor that is already black) result in no change and can therefore be disregarded. The other $2N - 2$ outcomes all result in a white square being colored black. If that is the white square numbered k, then the configuration that results is equivalent to a pair of strips like the original one, but with N replaced by $k - 1$ for one of the strips and by $N - k$ for the other, so that the expected number of white squares at the end of the process, given a particular value of k, will be $w(k - 1) + w(N - k)$. Of the $2N - 2$ outcomes that lead to a white square being colored black, one has $k = 1$, one has $k = N$, and the other values of k, with $2 \leq k \leq N - 1$, occur twice each. (The white squares at the ends are neighbors of a white square in just one way, the other white squares are adjacent to white squares on either side.) Because these $2N - 2$ outcomes are equally likely, we can conclude that

$$\begin{aligned} w(N) &= \frac{1}{2N-2}\big(w(0) + w(N-1) + 2(w(1) + w(N-2))+ \\ &\qquad\qquad \cdots + 2(w(N-2) + w(1)) + w(N-1) + w(0)\big) \\ &= \frac{1}{N-1}\Big(w(0) + w(N-1) + 2\sum_{n=1}^{N-2} w(n)\Big). \end{aligned}$$

Thus $(N-1)w(N) = w(N-1) + 2\sum\limits_{n=1}^{N-2} w(n)$. Subtracting this equation from the same one in which N is replaced by $N+1$, we get $Nw(N+1) - (N-1)w(N) = w(N) + w(N-1)$, that is,

$$w(N+1) = w(N) + \frac{1}{N}w(N-1). \qquad (*)$$

The solutions of this second-order linear homogeneous recurrence relation are known to be linear combinations of any two independent solutions. By inspection, one solution is $w(N) = N + 1$; we will use reduction of order to find the general solution.

Substituting $w(N) = (N+1)q(N)$ into $(*)$ yields $(N+2)q(N+1) = (N+1)q(N) + q(N-1)$, which can be rewritten as $(N+2)[q(N+1) - q(N)] = -[q(N) - q(N-1)]$. Thus if we let $\Delta(N) = q(N) - q(N-1)$, we have

$$\Delta(N+1) = -\frac{1}{N+2}\,\Delta(N),$$

which implies that

$$\Delta(N) = C\,\frac{(-1)^N}{(N+1)!}$$

for some constant C. Therefore,

$$q(N) = q(0) + \sum_{n=1}^{N} \Delta(n) = q(0) + C\left[\sum_{n=1}^{N} \frac{(-1)^n}{(n+1)!}\right],$$

from which we see that the general solution to $(*)$ is given by

$$w(N) = a \cdot (N+1) + C \cdot (N+1)\left[\sum_{n=1}^{N} \frac{(-1)^n}{(n+1)!}\right].$$

From the initial conditions we have $w(0) = 0$, so $a = 0$, and then $w(1) = 1$, so $C = -1$ and

$$w(N) = (N+1)\left[\sum_{j=2}^{N+1} \frac{(-1)^j}{j!}\right] = (N+1)\left[\sum_{j=0}^{N+1} \frac{(-1)^j}{j!}\right].$$

Finally,

$$\lim_{N\to\infty} \frac{w(N)}{N} = \lim_{N\to\infty}\left[\sum_{j=0}^{N+1} \frac{(-1)^j}{j!}\right] = \frac{1}{e}.$$

A5. Let a_n be the number of sets S of positive integers for which

$$\sum_{k\in S} F_k = n,$$

where the Fibonacci sequence $(F_k)_{k\geq 1}$ satisfies $F_{k+2} = F_{k+1} + F_k$
and begins $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$.
Find the largest integer n such that $a_n = 2020$.

Answer. $n = F_{4040} - 1$.

Solution 1. We will show that for any integer $m \geq 2$, the largest n such that $a_n = m$ is $n = F_{2m} - 1$; therefore, the answer is $F_{4040} - 1$.

Note that for every positive integer n, the inequalities $F_{2m-2} \leq n \leq F_{2m} - 1$ are satisfied for a unique $m \geq 2$. Thus it is enough to show the two following facts, each of which will be proved by induction on m:

a) For $n = F_{2m} - 1$, we have $a_n = m$.
b) Whenever $F_{2m-2} \leq n \leq F_{2m} - 1$, we have $a_n \geq m$.

Proof of a): For the base case $m = 2$, we have $n = F_4 - 1 = 2$ and the sets S with $\sum_{k\in S} F_k = n$ are $\{1, 2\}$ and $\{3\}$. Now let $n = F_{2m+2} - 1$. To get a decomposition $n = \sum_{k\in S} F_k$ of the desired form, we can add F_{2m+1} to any such decomposition of $F_{2m} - 1$; by the induction hypothesis, there are exactly m such. If a decomposition of n does *not* include F_{2m+1}, it must include all earlier F_k, because $n = F_{2m} + F_{2m-1} + \cdots + F_1$; this identity yields one additional decomposition, for a total of $m + 1$, completing the induction.

Proof of b): We'll use $m = 2$ and $m = 3$ as base cases, checking that $a_1 = 2$ (the sets are $\{1\}$ and $\{2\}$) and $a_2 = 2$ for $m = 2$, and that $a_3 = 3, a_4 = 3, a_5 = 3, a_6 = 4, a_7 = 3$ for $m = 3$. Now suppose that $F_{2m} \leq n \leq F_{2m+2} - 1$ and $m \geq 3$. Then we can write $n = F_q + \ell$, where $q = 2m$ or $q = 2m + 1$ and $0 \leq \ell \leq F_{q-1} - 1$. We distinguish three cases.

(1) If $0 \le \ell < F_{q-3}$, we can get a decomposition of n by adding F_q to any decomposition of ℓ (of which there are at least 2, unless $\ell = 0$, in which case we only have $n = F_q$) as well as by adding F_{q-1} to any decomposition of $F_{q-2} + \ell$ (which is a number less than F_{q-1}, so it cannot create repetition). By the induction hypothesis, there are at least m of the latter, so there are at least $m+1$ different decompositions of n, as desired.

(2) If $F_{q-3} \le \ell < F_{q-3} + F_{q-4} = F_{q-2}$, then we can get a decomposition of n by adding F_q to any decomposition of ℓ, and we can also get one by adding $F_{q-1} + F_{q-2}$ to any decomposition of ℓ. Because $F_{2m-4} \le \ell \le F_{2m} - 1$, by induction hypothesis there are at least $m-1$ decompositions of ℓ, so there are at least $2(m-1)$ decompositions of n. Because $m \ge 3$, we have $2(m-1) \ge m+1$, as desired.

(3) The final case occurs when $F_{q-2} \le \ell \le F_{q-1} - 1$. In this case, we can get a decomposition of n by adding F_q to any decomposition of ℓ; by the induction hypothesis, there are at least m such decompositions. We can also find at least one other decomposition by starting with $F_{q-1} + F_{q-2}$ and continuing to add the largest possible distinct Fibonacci number to keep the sum $\le n$, that is, by using the "greedy" algorithm. (This works because $F_q + \ell \le F_q + F_{q-1} - 1 = F_{q+1} - 1 = F_{q-1} + F_{q-2} + \cdots + F_1$, and if any Fibonacci number is not used, the sum of the remaining ones is greater than or equal to the one skipped, so the algorithm can continue until the sum reaches n.) So once more there are at least $m+1$ different decompositions of n, and the proof is complete.

Solution 2. We start by showing that every positive integer n can be written *uniquely* as a sum $n = \sum_{k \in S} F_k$ for which the set S contains no two consecutive integers and contains only integers that are at least 2. (This way of writing n will be referred to as the base-Fibonacci representation of n; the first few are $1 = F_2$, $2 = F_3$, $3 = F_4$, $4 = F_2 + F_4$.) To show that such a representation of n is possible, choose the largest m for which $F_m \le n$, and note that then $n < F_{m+1}$, so $n - F_m < F_{m-1}$. By induction, $n - F_m$ has a base-Fibonacci representation, for which the set of subscripts cannot include $m-1$, and adding F_m gives a base-Fibonacci representation for n. To show uniqueness of the representation, note that if we have such a representation $n = \sum_{k \in S} F_k$ and j is the largest element in S, then

$$\begin{aligned} n &\le F_j + F_{j-2} + F_{j-4} + \cdots \\ &= (F_{j+1} - F_{j-1}) + (F_{j-1} - F_{j-3}) + \cdots \\ &= F_{j+1} - 1, \end{aligned}$$

so j must be the largest integer with $F_j \le n$; then $n - F_j = \sum_{k \in S - \{j\}} F_k$ is a base-Fibonacci representation, which is unique by induction.

Next, we identify any nonempty finite set S of positive integers with the finite sequence $(s_k)_{k \ge 1}$ of zeros and ones, ending with a 1, such that $s_k = 1 \Leftrightarrow k \in S$. In particular, we can then define

$$f(S) = \sum s_k F_k = \sum_{k \in S} F_k,$$

and we are looking for the largest n such that there are exactly 2020 sets S with $f(S) = n$. We will show, more generally, that the largest n for which there are exactly $m \ge 2$ sets with $f(S) = n$ is $n = F_{2m} - 1$, so that the answer is $F_{4040} - 1$.

To begin, note that for any positive integer n, one set S_0 with $f(S_0) = n$ is given by the base-Fibonacci representation of n, and any other set S with $f(S) = n$ can be transformed into S_0 as follows. Replace any occurrence of consecutive terms $(1,1,0)$ in the sequence by $(0,0,1)$; if there is no occurrence of $(1,1,0)$ but the sequence starts with $(1,0)$, replace that start by $(0,1)$. Repeat these "moves" until no further move is possible, at which point we must have arrived at S_0, because there are no longer consecutive 1's in the sequence and the sequence starts with 0. Therefore, we can count the number of sets with $f(S) = n$ by counting the number of sets that can be transformed into S_0 by a sequence of such moves, or equivalently the number of sets that can be obtained from S_0 (including S_0 itself) by reversing these moves. These reverse moves send $(0,0,1)$ anywhere in the sequence to $(1,1,0)$ (we'll refer to this as an A move) or $(0,1)$ at the beginning of the sequence to $(1,0)$ (a B move).

Suppose that $n = F_{2m} - 1$. Then the base-Fibonacci representation of n is

$$n = F_3 + F_5 + \cdots + F_{2m-3} + F_{2m-1}\,,$$

corresponding to the sequence $0,0,1,0,1,0,1,\ldots,0,1$, and when we start reversing the moves we see that at every step there is only one choice, which is an A move; after k steps we will have

$$n = F_1 + F_2 + \cdots + F_{2k} + F_{2k+3} + F_{2k+5} + \cdots + F_{2m-3} + F_{2m-1}$$

and we get such representations for $k = 0,1,\ldots,m-1$, so there are exactly m sets S with $f(S) = n$.

To finish the proof, we now show by induction that if $n \geq F_{2m}$, there are more than m sets S with $f(S) = n$. Note that because $n \geq F_{2m}$, the base-Fibonacci representation of n corresponds to a sequence which has a 1 in, or to the right of, the $2m$th position. Thus it is enough to prove that for any base-Fibonacci sequence S_0 with its final 1 in either the $2m$th position or the $(2m+1)$st position, there are at least $m+1$ different outcomes (counting S_0 itself) of the A and B moves described above. We will do so by induction, using the rightmost string of at least two successive zeros that occurs in S_0 before the final 1. If there is no such string of zeros, then S_0 must be precisely of the form $0,1,0,1,0,1,\ldots,0,1$ with its final 1 in the $2m$th position; in this case we can start with a B move and then make up to $m-1$ A moves, so there are in fact exactly $m+1$ different outcomes. If the rightmost string of at least two successive zeros actually comes at the very beginning of the sequence, say that the sequence starts with exactly z zeros ($z \geq 2$), so it consists of those z zeros followed by $1,0,1,0,\ldots,1,0,1$, say $(r+1)$ ones and r zeros. Then we can start with $\lfloor z/2 \rfloor$ A moves at the beginning of the sequence; if z is odd, we can follow those up with a B move, so, whether z is even or odd, we have a total of $\lfloor (z+1)/2 \rfloor$ moves available at the beginning of the sequence. After that we have an additional r A moves, as each 1 that has not been moved yet can be "pushed to the left" (using an A move) in its turn. In all, we have at least $\lfloor (z+1)/2 \rfloor + r + 1$ different outcomes (counting S_0 itself); meanwhile, the length of the sequence S_0 is $z+2r+1$. Whether this equals $2m$ or $2m+1$, we have $\lfloor (z+1)/2 \rfloor + r + 1 = m + 1$, so we are done in this case. We are left with the case that the rightmost string of at least two zeros in S_0 does not come at the beginning of the sequence; say it comes after a 1 in the ath position and consists of exactly z zeros, followed by $(r+1)$ ones and r zeros, alternating as in the previous case. By the induction hypothesis, there are at least $\lfloor a/2 \rfloor + 1$ different outcomes available (including the starting "state") for just the first a positions of the sequence. For each of these "partial" outcomes, the rest of the sequence, starting with the z consecutive zeros, can be treated as in the previous case, except that if z is odd, we do not have the follow-up B move available after the initial $\lfloor z/2 \rfloor$ A moves. Thus we have at least $\lfloor z/2 \rfloor + r + 1$ different possible outcomes for the part of the sequence after the first a positions, so overall we have at least

$$\begin{aligned}(\lfloor a/2\rfloor+1)(\lfloor z/2\rfloor+r+1) &\geq \lfloor a/2\rfloor+\lfloor z/2\rfloor+r+1+\lfloor a/2\rfloor\lfloor z/2\rfloor\\ &\geq \lfloor a/2\rfloor+\lfloor z/2\rfloor+r+2\end{aligned}$$

outcomes, because $a \geq 2$ and $z \geq 2$. The length of the sequence is $a+z+2r+1$. If this equals $2m$, then one of a and z is even (and the other is odd), so

$$\lfloor a/2\rfloor+\lfloor z/2\rfloor+r+2 = a/2+z/2-1/2+r+2=(a+z+2r+3)/2=m+1\,;$$

if the length equals $2m+1$, then a and z have the same parity and

$$\lfloor a/2\rfloor+\lfloor z/2\rfloor+r+2 \geq a/2+z/2-1+r+2=(a+z+2r+2)/2=m+1\,.$$

This estimate concludes the proof.

A6. For a positive integer N, define the function

$$f_N(x)=\sum_{n=0}^{N}\frac{N+1/2-n}{(N+1)(2n+1)}\sin\big((2n+1)x\big)\,.$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.

Answer. $M=\dfrac{\pi}{4}$.

Solution. Note that $f_N(x)$ is an odd function with period 2π. The following computation allows us to write its derivative in closed form:

$$\begin{aligned}
f_N'(x) &= \sum_{n=0}^{N}\frac{2N+1-2n}{2(N+1)}\cos((2n+1)x)=\sum_{n=0}^{N}\frac{2N+1-2n}{2(N+1)}\ \mathrm{Re}\left(e^{(2n+1)ix}\right)\\
&=\frac{1}{2(N+1)}\ \mathrm{Re}\left(e^{2(N+1)ix}\sum_{n=0}^{N}(2N+1-2n)e^{(2n-1-2N)ix}\right)\\
&=\frac{1}{2(N+1)}\ \mathrm{Re}\left(ie^{2(N+1)ix}\frac{d}{dx}\sum_{n=0}^{N}e^{(2n-1-2N)ix}\right)\\
&=\frac{1}{2(N+1)}\ \mathrm{Re}\left(ie^{2(N+1)ix}\frac{d}{dx}\left(e^{(-1-2N)ix}\frac{1-e^{2(N+1)ix}}{1-e^{2ix}}\right)\right)\\
&=\frac{1}{2(N+1)}\ \mathrm{Re}\left(ie^{2(N+1)ix}\frac{d}{dx}\frac{1-e^{-2(N+1)ix}}{e^{ix}-e^{-ix}}\right)\\
&=\frac{1}{4(N+1)}\ \mathrm{Re}\left(e^{2(N+1)ix}\frac{d}{dx}\frac{1-e^{-2(N+1)ix}}{\sin x}\right)\\
&=\frac{1}{4(N+1)}\ \mathrm{Re}\left(\frac{2(N+1)i}{\sin x}+\frac{(1-e^{2(N+1)ix})\cos x}{\sin^2 x}\right)\\
&=\frac{[1-\cos(2(N+1)x)]\cos x}{4(N+1)\sin^2 x}=\frac{\sin^2((N+1)x)}{2(N+1)\sin^2 x}\cos x\,.
\end{aligned}$$

In particular, the derivative has the same sign as $\cos x$. (This is still true where $\sin x = 0$, because then $x = k\pi$ for some integer k, and for all $0 \leq n \leq N$, $\cos((2n+1)k\pi) = \cos k\pi$, so that the first expression for $f_N'(x)$ above is a positive multiple of $\cos x$. Alternatively, one can use continuity of the derivative and l'Hôpital's rule.) It follows that $f_N(x)$ has its maximum value for $x = \pi/2$. That value is

$$
\begin{aligned}
f_N\left(\frac{\pi}{2}\right) &= \sum_{n=0}^{N} \frac{2N+1-2n}{(2N+2)(2n+1)} \cdot (-1)^n \\
&= \sum_{n=0}^{N} \left(\frac{1}{2n+1} - \frac{1}{2N+2}\right) \cdot (-1)^n \\
&= \begin{cases} \left(\sum\limits_{n=0}^{2M} \dfrac{(-1)^n}{2n+1}\right) - \dfrac{1}{4M+2} & \text{when } N = 2M \text{ is even} \quad \text{and} \\[2ex] \left(\sum\limits_{n=0}^{2M} \dfrac{(-1)^n}{2n+1}\right) - \dfrac{1}{4M+3} & \text{when } N = 2M+1 \text{ is odd.} \end{cases}
\end{aligned}
$$

From here it is straightforward to check that $f_{2M}(\pi/2) - f_{2M-1}(\pi/2) = \frac{1}{(4M+1)(4M+2)}$ and $f_{2M+1}(\pi/2) - f_{2M}(\pi/2) = \frac{1}{(4M+2)(4M+3)}$, so the maximum value $f_N(\pi/2)$ is an increasing function of N. Thus the least upper bound on $f_N(x)$ that is valid for all N and x is

$$
\lim_{N\to\infty} f_N\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}\,.
$$

Variant. An alternate computation of the derivative $f_N'(x)$ uses the trigonometric sums given by the
Lemma:

$$
\sum_{n=0}^{N} \cos[(2n+1)x] = \frac{\sin[(N+1)x]\cos[(N+1)x]}{\sin x}\,, \qquad \sum_{n=0}^{N} \sin[(2n+1)x] = \frac{\sin^2[(N+1)x]}{\sin x}\,.
$$

Proof: Let $z = e^{ix}$, and note that

$$
\begin{aligned}
\sum_{n=0}^{N} e^{i(2n+1)x} &= \sum_{n=0}^{N} z^{2n+1} = \frac{z - z^{2N+3}}{1-z^2} \\
&= \frac{z^{2N+2}-1}{z-1/z} = z^{N+1}\left(\frac{z^{N+1} - z^{-(N+1)}}{z-1/z}\right) \\
&= \left(\cos[(N+1)x] + i\sin[(N+1)x]\right)\frac{\sin[(N+1)x]}{\sin x}.
\end{aligned}
$$

Taking real and imaginary parts, we get the desired sums, proving the lemma.

We now use the identity

$$
\frac{N+1/2-n}{(N+1)(2n+1)} = \frac{1}{2n+1} - \frac{1}{2(N+1)}
$$

to split $f_N(x)$ into two pieces:

$$f_N(x) = \sum_{n=0}^{N} \frac{\sin[(2n+1)x]}{2n+1} - \frac{1}{2(N+1)} \sum_{n=0}^{N} \sin[(2n+1)x]$$
$$= \sum_{n=0}^{N} \frac{\sin[(2n+1)x]}{2n+1} - \frac{1}{2(N+1)} \frac{\sin^2[(N+1)x]}{\sin x} \,,$$

using the second part of the lemma. The derivative is therefore

$$f_N'(x) = \sum_{n=0}^{N} \cos[(2n+1)x] - \frac{1}{2(N+1)} \left(\frac{2(N+1)\sin[(N+1)x]\cos[(N+1)x]}{\sin x} - \frac{\sin^2[(N+1)x]}{\sin^2 x} \cdot \cos x \right)$$
$$= \frac{\sin^2((N+1)x)}{2(N+1)\sin^2 x} \cos x \,,$$

because the first two summands on the right cancel by the first part of the lemma.