B1. For a positive integer n, define $d(n)$ as the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

Answer. 1990.

Solution 1. We will show that if we let $d(0) = 0$, start the sum at $k = 0$ (which does not change its value), and group the terms from that start in groups of sixteen each, then each complete group contributes 0 to the sum. Therefore, S is equal to the sum starting at $2016 = 16 \cdot 126$, that is,

$$S = \sum_{k=2016}^{2020} (-1)^{d(k)} k^3.$$

From the binary expansion 11, 111, 100, 000 of 2016, we observe that
$d(2016) = 6,\ d(2017) = 7,\ d(2018) = 7,\ d(2019) = 8,\ d(2020) = 7$ and so

$$\begin{aligned} S &= (2016)^3 - (2017)^3 - (2018)^3 + (2019)^3 - (2020)^3 \\ &\equiv (-4)^3 - (-3)^3 - (-2)^3 + (-1)^3 = -64 + 27 + 8 - 1 = -30 \equiv 1990 \pmod{2020}. \end{aligned}$$

It remains to prove the desired cancellation, which results from the polynomial $P(x) = x^3$ having degree 3 and from $2^{3+1} = 16$. In fact, we can apply the following lemma.

Lemma: Suppose $P(x)$ is a polynomial of degree $n \geq 0$, let $N = 2^{n+1} - 1$, and let $m \geq 0$ be any integer. Then

$$Q_m = \sum_{j=0}^{N} (-1)^{d\left(m2^{n+1}+j\right)}\ P(m2^{n+1} + j) = 0.$$

Proof: Note that for $0 \leq j \leq N$, there are no carries in binary in the addition $m2^{n+1} + j$, so $d\left(m2^{n+1} + j\right) = d\left(m2^{n+1}\right) + d(j)$ and we have $Q_m = (-1)^{d\left(m2^{n+1}\right)} q_m(0)$, where

$$q_m(x) = \sum_{j=0}^{N} (-1)^{d(j)} p_m(x + j)\,,\ p_m(x) = P(m2^{n+1} + x).$$

Now $p_m(x)$, a translate of $P(x)$, is again a polynomial of degree n, and so it is enough to show that if $p(x)$ is any polynomial of degree n and $q(x) = \sum\limits_{j=0}^{N} (-1)^{d(j)} p(x + j)$, then $q(0) = 0$. In fact, we will see that $q(x)$ is identically zero. Define the difference operator Δ_k on polynomials by $\Delta_k R(x) = R(x + k) - R(x)$. We then have

$$\begin{aligned} q(x) &= \sum_{j=0}^{2^n-1} (-1)^{d(j)} p(x + j) + \sum_{j=2^n}^{N} (-1)^{d(j)} p(x + j) \\ &= \sum_{j=0}^{2^n-1} \left[(-1)^{d(j)} p(x + j) + (-1)^{d(j+2^n)} p(x + j + 2^n)\right] \\ &= -\Delta_{2^n} \sum_{j=0}^{2^n-1} (-1)^{d(j)} p(x + j) \end{aligned}$$

because $d(j+2^n) = d(j)+1$ for $0 \le j \le 2^n - 1$. By continuing to halve the interval for j in this way we end up with

$$q(x) = (-1)^{n+1}\Delta_{2^n} \cdots \Delta_4\Delta_2\Delta_1 p(x).$$

However, each application of a difference operator Δ_k lowers the degree of a nonconstant polynomial by 1, so $\Delta_{2^{n-1}} \cdots \Delta_4\Delta_2\Delta_1 p(x)$ is constant and $q(x)$ is identically zero, completing the proof of the lemma.

Solution 2. Let $\overline{S}_{n,q} = \sum\limits_{k=0}^{n} (-1)^{d(k)} k^q \bmod 2020$, so we are looking for $\overline{S}_{2020,3}$. Note that $d(2k) = d(k)$ and $d(2k+1) = d(k)+1$, so splitting the sum into even and odd terms we get

$$\begin{aligned}
\overline{S}_{2020,3} &\equiv \sum_{k=0}^{1009}(-1)^{d(2k)}(2k)^3 + \sum_{k=0}^{1009}(-1)^{d(2k+1)}(2k+1)^3 \\
&= \sum_{k=0}^{1009}(-1)^{d(k)}\big(8k^3 - 8k^3 - 12k^2 - 6k - 1\big) \\
&\equiv -12\overline{S}_{1009,2} - 6\overline{S}_{1009,1} - \overline{S}_{1009,0}\,,
\end{aligned}$$

where the congruences are modulo 2020. Similarly, we have

$$\begin{aligned}
\overline{S}_{1009,2} &\equiv \sum_{k=0}^{504}(-1)^{d(2k)}(2k)^2 + \sum_{k=0}^{504}(-1)^{d(2k+1)}(2k+1)^2 \\
&= \sum_{k=0}^{504}(-1)^{d(k)}\big(4k^2 - 4k^2 - 4k - 1\big) \\
&\equiv -4\overline{S}_{504,1} - \overline{S}_{504,0}
\end{aligned}$$

and

$$\begin{aligned}
\overline{S}_{1009,1} &\equiv \sum_{k=0}^{504}(-1)^{d(2k)}(2k) + \sum_{k=0}^{504}(-1)^{d(2k+1)}(2k+1) \\
&= \sum_{k=0}^{504}(-1)^{d(k)}(-1) \\
&\equiv -\overline{S}_{504,0}\,.
\end{aligned}$$

Combining the results so far, we have

$$\begin{aligned}
\overline{S}_{2020,3} &\equiv -12(-4\overline{S}_{504,1} - \overline{S}_{504,0}) + 6\overline{S}_{504,0} - \overline{S}_{1009,0} \\
&\equiv 48\overline{S}_{504,1} + 18\overline{S}_{504,0} - \overline{S}_{1009,0}\,.
\end{aligned}$$

Note that

$$\begin{aligned}
\overline{S}_{504,1} &\equiv \sum_{k=0}^{252}(-1)^{d(2k)}(2k) + \sum_{k=0}^{251}(-1)^{d(2k+1)}(2k+1) \\
&= (-1)^{d(504)}(504) + \sum_{k=0}^{251}(-1)^{d(k)}(-1) \\
&\equiv (-1)^{d(504)}(504) - \overline{S}_{251,0} \\
&= 504 - \overline{S}_{251,0}\,,
\end{aligned}$$

because the binary expansion of 504 is 111111000, so that $d(504) = 6$ is even. Finally, $\overline{S}_{n,0} = 0$ whenever n is odd (using the same splitting into even and odd terms), so $\overline{S}_{503,0} = 0$ and $\overline{S}_{504,0} = (-1)^{d(504)} = 1$. It follows that

$$\begin{aligned}
\overline{S}_{2020,3} &\equiv 48(504 - 0) + 18 \cdot 1 - 0 \\
&= 24210 \equiv 1990.
\end{aligned}$$

B2. Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move, and therefore loses. For what values of n and k does Alice have a winning strategy?

Answer. Alice has a winning strategy if and only if at least one of k and n is odd.

Solution. Number the holes, from left to right, $1, 2, \ldots, n$. We first show that when k and n are both even, Bob has a winning strategy. In this case we can divide the holes into disjoint adjacent pairs $P_i = \{2i-1, 2i\}$ with $1 \leq i \leq n/2$. At the beginning of the game the pegs completely occupy the holes in the leftmost $k/2$ pairs, and all the holes in the remaining pairs are vacant. Thus Alice's first move must take a peg from an occupied pair of holes and place it in one of a vacant pair of holes. A winning strategy for Bob is to always take the other peg of the pair that Alice moved from and place it in the remaining hole of the pair that Alice moved to. Thus after each of Bob's moves, each of the pairs P_i either has pegs in both holes or in neither, whereas after each of Alice's moves, there are two of the pairs P_i with one peg each. In particular, Alice can never reach the ending position, and the game will end after one of Bob's moves.

If k and n are not both even, Alice always has a first move available which will leave Bob either with no moves at all, or with a position equivalent to the starting position of our game with even integers k_1 and n_1, $1 \leq k_1 < n_1$. Thus by the case discussed in the previous paragraph, Alice (as the second player from that position) has a winning strategy. Specifically, if k and n are both odd, Alice can move the peg in hole k to hole n, leaving $k_1 = k-1$ pegs at the beginning of a line of $n_1 = n-1$ remaining holes. (If $k = 1$, the game is then over.) If k is odd and n is even, Alice can move the peg in hole 1 to hole n, winning the game immediately if $k = 1$ and otherwise leaving $k_1 = k-1$ pegs at the beginning of a line of $n_1 = n-2$ remaining holes. Finally, if k is even and n is odd, Alice can move the peg in hole 1 to hole $k+1$, winning the game immediately if $n = k+1$ and otherwise leaving $k_1 = k$ pegs at the beginning of a line of $n_1 = n-1$ remaining holes. In each of these three cases, after making the indicated first move, Alice can use Bob's strategy from the previous paragraph to win.

Comment. The game with k pegs and n holes is equivalent to the game with $n-k$ pegs and n holes (moving the k pegs to the right is equivalent to moving the $n-k$ vacant spaces to the left). This symmetry can be used to reduce the three cases considered in the second paragraph to just two.

B3.

Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ.

Answer. The expected value is $1 + \ln(1/\delta)$.

Solution 1. Let $\rho_n(x)$ be the probability density for the location of x_n. Note that $0 \le x_n \le 1$ for all n, so these density functions all have support $[0,1]$. They can be found recursively from $\rho_1(x) = 1$ and

$$\rho_{n+1}(x) = \int_{y=x}^{1} \rho_n(y)\,\frac{dy}{y}\,.$$

This yields

$$\rho_2(x) = \int_{y=x}^{1} \frac{dy}{y} = -\ln(x), \quad \rho_3(x) = \int_{y=x}^{1} (-\ln y)\,\frac{dy}{y} = \frac{[-\ln(x)]^2}{2}\,,$$

which suggests that in general

$$\rho_n(x) = \frac{[-\ln(x)]^{n-1}}{(n-1)!}\,;$$

this is straightforward to check by induction.

Let q_n be the probability that $x_n < \delta$ but $x_{n-1} \ge \delta$, that is, the probability that $Z = n$. Then $q_1 = \delta$, and for $n \ge 2$ we have

$$\begin{aligned} q_n &= \int_0^{\delta} \rho_n(x) - \rho_{n-1}(x)\,dx \\ &= \int_0^{\delta} \frac{[-\ln(x)]^{n-1}}{(n-1)!} - \frac{[-\ln(x)]^{n-2}}{(n-2)!}\,dx \\ &= \left.\frac{x[-\ln(x)]^{n-1}}{(n-1)!}\right|_0^{\delta} \\ &= \frac{\delta[-\ln(\delta)]^{n-1}}{(n-1)!}\,. \end{aligned}$$

Finally, the expected value of Z is

$$\begin{aligned}
E(Z) &= \sum_{n=1}^{\infty} n q_n \\
&= \delta + \sum_{n=2}^{\infty} n \frac{\delta[-\ln(\delta)]^{n-1}}{(n-1)!} \\
&= \sum_{m=0}^{\infty} (m+1) \frac{\delta[-\ln(\delta)]^{m}}{m!} \\
&= \sum_{m=0}^{\infty} \frac{\delta[-\ln(\delta)]^{m}}{m!} + \sum_{m=1}^{\infty} \frac{\delta[-\ln(\delta)]^{m}}{(m-1)!} \\
&= \delta \exp[-\ln(\delta)] - \ln(\delta) \cdot \delta \exp[-\ln(\delta)] \\
&= 1 + \ln(1/\delta).
\end{aligned}$$

Solution 2. A short calculation shows that if X is a uniform random variable on $[0,1]$, then $U = -\ln X$ is an exponential random variable with expected value $\lambda = 1$, and probability density function $p_U(t) = e^{-t}$ for $t \geq 0$. Note that this applies to each of $X_1 = x_1/x_0$, $X_2 = x_2/x_1$, $\ldots$, $X_n = x_n/x_{n-1}$, and that the product $X_1 X_2 \cdots X_n$ equals x_n . Thus if $U_1, U_2, \ldots, U_n, \ldots$ are the corresponding exponential random variables, the problem is equivalent to finding the expected number $Z = Z(D)$ of i.i.d. samples that must be taken to get $U_1 + \cdots + U_Z > D$, where $D = -\ln \delta = \ln(1/\delta)$. By "First-Step Analysis" (considering what the situation is after the first sample) we see that

$$\begin{aligned}
Z(D) &= 1 + \int_0^D Z(D-t) p_U(t)\, dt = 1 + \int_0^D Z(D-t) e^{-t}\, dt \\
&= 1 + e^{-D} \int_0^D Z(u) e^{u}\, du\,.
\end{aligned}$$

Differentiating with respect to D gives

$$\begin{aligned}
Z'(D) &= -e^{-D} \int_0^D Z(u) e^{u}\, du + e^{-D} Z(D) e^{D} \\
&= -(Z(D) - 1) + Z(D) = 1,
\end{aligned}$$

where the second line follows by substituting for the integral using the First-Step Analysis equation. Thus $Z(D) = Z(0) + D = 1 + D = 1 + \ln(1/\delta)$.

Solution 3. Note that given $x_{k-1} \geq \delta$, the probability that x_k is not smaller than δ is $(x_{k-1} - \delta)/x_{k-1} \leq 1 - \delta$. Hence the probability that $Z = n$ is bounded above by $(1-\delta)^{n-1}$. Thus the expected value of Z is bounded by the sum of the convergent series $\sum_{n=1}^{\infty} n(1-\delta)^{n-1}$, and thus is finite.

Let $f(\delta)$ be this expected value, as a function of δ. Note that this function is monotone decreasing. If after one step of the iteration we are at $x_1 \geq \delta$, then rescaling by a factor $1/x_1$, we see that we have essentially returned to the original problem but with δ replaced by δ/x_1 . Thus

$$f(\delta) = 1 + \int_\delta^1 f(\delta/x)\, dx.$$

Letting $g(t) = f(1/t)$ and making the substitution $u = tx$, this becomes

$$g(t) = 1 + \frac{1}{t}\int_1^t g(u)\,du.$$

Since f is monotone decreasing, g is monotone increasing and hence integrable. Thus it follows from this functional equation that g is continuous for $t > 0$. Hence the integral in the functional equation is a differentiable function of t, and it follows that g is differentiable.

Multiplying both sides of the functional equation by t and then taking the derivative of both sides leads to

$$g(t) + tg'(t) = 1 + g(t), \quad \text{so} \quad tg'(t) = 1.$$

Integrating and using the initial condition $g(1) = 1$, we get $g(t) = 1 + \ln t$ and hence $f(\delta) = 1 + \ln(1/\delta)$.

B4. Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\dfrac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$.

Evaluate $M(2020)$.

Answer. $\dfrac{1}{4040}$.

Solution. We will show that $M(n) = \dfrac{1}{2n}$ for all n, by partitioning V_n into subsets such that the average of $\dfrac{1}{q(\mathbf{v})}$ over each subset is $\dfrac{1}{2n}$. First note that giving an element $\mathbf{v} \in V_n$ is equivalent to giving a sequence of length $2n$ consisting of symbols U (for "up") and D (for "down") so that each symbol occurs n times in the sequence; the symbol in position i is U or D according to whether $s_i - s_{i-1}$ is 1 or -1. With this representation of elements of V_n, there is a natural "cyclic rearrangement" map $\sigma : V_n \to V_n$ which moves each of the symbols one position back cyclically, that is, the symbol in position 1 moves to position $2n$, and for every $j > 1$ the symbol in position j moves to position $j-1$. In terms of the $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$, this works out to

$$\sigma(\mathbf{v}) = (t_0, t_1, \cdots, t_{2n-1}, t_{2n}) \quad \text{where} \quad t_j = s_{j+1} - s_1\,,$$

with the understanding that subscripts are taken modulo $2n$. (Note that $t_0 = t_{2n} = 0$ and that $|t_j - t_{j-1}| = |s_{j+1} - s_j| = 1$.)

From the representation using the symbols U and D, we see that $\sigma^{2n}(\mathbf{v}) = \mathbf{v}$. In particular, for any $\mathbf{v} \in V_n$, the list of elements $\mathbf{v}, \sigma(\mathbf{v}), \sigma^2(\mathbf{v}), \ldots, \sigma^{2n-1}(\mathbf{v})$ runs through the orbit under σ of $\mathbf{v}$ a whole number of times. So the average of $\dfrac{1}{q(\mathbf{w})}$ for $\mathbf{w}$ on that list of elements is the same as the average over the orbit of $\mathbf{v}$; because the orbits partition V_n, it is enough to show that this average is $\dfrac{1}{2n}$ for any $\mathbf{v}$.

Now note that

$$\frac{1}{q(\sigma(\mathbf{v}))} = \frac{1}{1+\sum\limits_{j=1}^{2n-1} 3^{s_{j+1}-s_1}} = \frac{3^{s_1}}{3^{s_1}+\sum\limits_{j=1}^{2n-1} 3^{s_{j+1}}} = \frac{3^{s_1}}{q(\mathbf{v})}$$

because $3^{s_{2n}} = 1$. Applying this with $\mathbf{v}$ replaced by $\sigma(\mathbf{v})$ yields

$$\frac{1}{q(\sigma^2(\mathbf{v}))} = \frac{3^{s_2-s_1}}{q(\sigma(\mathbf{v}))} = \frac{3^{s_2}}{q(\mathbf{v})}$$

and similarly, by induction on j,

$$\frac{1}{q(\sigma^j(\mathbf{v}))} = \frac{3^{s_j}}{q(\mathbf{v})}.$$

To average $\dfrac{1}{q(\mathbf{w})}$ over the list $\mathbf{v}, \sigma(\mathbf{v}), \sigma^2(\mathbf{v}), \ldots, \sigma^{2n-1}(\mathbf{v})$, we add these answers for $j = 0, 1, \ldots, 2n-1$ and divide by $2n$. But the sum of these answers is $\dfrac{q(\mathbf{v})}{q(\mathbf{v})} = 1$, so we are done.

B5. For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that $3 - z_1 - z_2 - z_3 - z_4 + z_1z_2z_3z_4 \neq 0$.

Solution 1. Let $e_k(Z)$ denote the kth elementary symmetric function of $Z := (z_1, z_2, z_3, z_4)$, so that we want to show $3 - e_1(Z) + e_4(Z) \neq 0$. We will transform the variables first by $z_j = 1 - y_j$ and then by $y_j = 1/w_j$. The condition $z_j \neq 1$ becomes $y_j \neq 0$, so that w_j is indeed defined, while the condition $|z_j| = 1$ implies $\mathrm{Re}(w_j) = 1/2$. Meanwhile, using similar notation for the elementary symmetric functions of the y's and the w's, we find that

$$3 - e_1(Z) + e_4(Z) = e_2(Y) - e_3(Y) + e_4(Y) = \frac{1 - e_1(W) + e_2(W)}{e_4(W)}.$$

Now let

$$w_j = \frac{1}{2} + i\, v_j\,.$$

Then

$$w_j w_k = \frac{1}{4} - v_j v_k + \frac{i}{2}(v_j + v_k),$$

so for the symmetric functions we have

$$e_1(W) = \sum w_j = 2 + i\, e_1(V),\ \ e_2(W) = \sum_{j<k} w_j w_k = \frac{3}{2} - e_2(V) + \frac{3i}{2} e_1(V)$$

and it is enough to show that

$$1 - e_1(W) + e_2(W) = \frac{1}{2} + \frac{i}{2} e_1(V) - e_2(V)$$

is never zero for real $V = (v_1, v_2, v_3, v_4)$.

If this quantity were zero, taking real and imaginary parts we would have $e_1(V) = 0$, $e_2(V) = 1/2$. However, because V is real we have

$$e_1(V)^2 = \Big(\sum v_j\Big)^2 = \sum v_j^2 + 2\sum_{j<k} v_j v_k \geq 2e_2(V),$$

so those values for $e_1(V)$ and $e_2(V)$ are impossible.

Solution 2. We use a bilinear (linear fractional) transformation to map the circle $|z| = 1$ to the real line. Because $z_j \neq 1$, it seems natural to map 1 to the point at infinity; a transformation that will do these things is given by

$$w = i\,\frac{1+z}{1-z} \qquad \Leftrightarrow \qquad z = \frac{w-i}{w+i}\ .$$

To check, $|z| = 1$ implies $|w - i| = |w + i|$, from which it follows that w is real.

By (a significant amount of) direct computation we find that

$$\begin{aligned}&3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \\ &\quad = \frac{8 - 4i(w_1 + w_2 + w_3 + w_4) - 4(w_1w_2 + w_1w_3 + w_1w_4 + w_2w_3 + w_2w_4 + w_3w_4)}{(w_1 + i)(w_2 + i)(w_3 + i)(w_4 + i)}.\end{aligned}$$

If this were zero for real numbers w_i, taking real and imaginary parts of the numerator we would get

$8-4(w_1w_2+w_1w_3+w_1w_4+w_2w_3+w_2w_4+w_3w_4) = 0$ and $w_1+w_2+w_3+w_4 = 0$, respectively.

However,

$$\begin{aligned}&8 - 4(w_1w_2 + w_1w_3 + w_1w_4 + w_2w_3 + w_2w_4 + w_3w_4) \\ &\qquad = 8 + 2\left[w_1^2 + w_2^2 + w_3^2 + w_4^2 - (w_1 + w_2 + w_3 + w_4)^2\right]\end{aligned}$$

is always positive when $w_1 + w_2 + w_3 + w_4 = 0$, completing the proof.

B6. Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1)\rfloor} \geq 0\,.$$

(As usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

Solution. Let $\alpha = \sqrt{2} - 1$; note that this irrational number has the crucial property

$$\frac{1}{\alpha} = \sqrt{2} + 1 = 2 + \alpha.$$

Also, let $a_k = (-1)^{\lfloor k\alpha\rfloor}$. The sequence (a_k) is formed by the terms of the series whose partial sums we are looking at, and it consists of "signs" ± 1. Because $1/3 < \alpha < 1/2$, the signs come in runs of two or three equal signs, starting with a run of two 1s because $\lfloor \alpha\rfloor = \lfloor 2\alpha\rfloor = 0$ and $\lfloor 3\alpha\rfloor = 1$. Now suppose we omit two of the signs from each run, so the runs of two equal signs are deleted altogether and each run of three equal signs is replaced by a single sign. Denote the new sequence of signs by (b_k); that is, b_k is the value taken by the k-th run of length 3 in the sequence (a_k).

We will show below that $a_k = b_k$. Assuming this for now, we can prove the desired result by a reduction argument, as follows. Suppose that the result is false, and let N be the least value of n for which $\sum_{k=1}^{n} a_k < 0$. Then a_{N-2}, a_{N-1}, a_N must be a run of three -1s, because every run of fewer -1s is preceded by a run of 1s of at least equal length, so that omitting both those runs can only decrease the partial sum. Suppose that up to and including this point, there are m runs of length 3. If we pass from the sequence (a_k) to the sequence (b_k), we delete two entries from each run, starting with two 1s and ending with two -1s, so the sum of all the terms will be unchanged. That is,

$$\sum_{k=1}^{m} b_k = \sum_{k=1}^{N} a_k < 0\,,$$

and because $a_k = b_k$ we have $\sum_{k=1}^{m} a_k < 0$. But $m \leq N/3$, contradicting the minimality of N.

It remains only to prove that $a_k = b_k$. Suppose that the k-th run of length 3 is the $(t+1)$-st run overall, that is, the run for which the floor is equal to t. Then $b_k = (-1)^t$. There are $2t+(k-1)$ terms before this run, so it consists of the terms with subscripts $2t+k, 2t+k+1$, and $2t+k+2$, and we have the inequalities

$$t < (2t+k)\alpha\,,\ (2t+k+2)\alpha < t+1\,.$$

Isolating k in each of these, we get

$$k > t(\frac{1}{\alpha} - 2) = t\alpha\,,\ k < (t+1)(\frac{1}{\alpha} - 2) = (t+1)\alpha\,,$$

so $t\alpha < k < (t+1)\alpha$ and thus $\lfloor k/\alpha \rfloor = t$. But $k/\alpha = k(2+\alpha) = 2k + k\alpha$, so $\lfloor k\alpha \rfloor = t - 2k$, $a_k = (-1)^{\lfloor k\alpha \rfloor} = (-1)^{t-2k} = (-1)^t = b_k$, and we are done.