Carl B. Allendoerfer Awards

Jordan Bell and Viktor Blåsjö

"Pietro Mengoli's 1650 Proof that the Harmonic Series Diverges," *Mathematics Magazine*, 91:5, 341–347, 10.1080/0025570X.2018.1506656.

Everyone who's taken enough calculus has seen at least one proof that the harmonic series diverges. Fewer know that Pietro Mengoli gave the first published proof of this divergence, in 1650. The result had been known for centuries, but Mengoli's argument had a distinctive flavor that merits a new look—particularly because modern accounts have not always faithfully relayed his methods. The authors take the reader on a journey not only through Mengoli's arguments, but through his actual words, by providing a complete English translation of the proof.

Mengoli's proof proceeds by grouping terms into blocks of three and using the inequality

$$\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}>\frac{3}{n}.$$

At this point it is tempting from our modern perspective to argue that we obtain a lower bound for the sum of the harmonic series in terms of the sum itself, and no finite quantity can satisfy such a bound. Indeed, this is what several modern accounts claim Mengoli's proof does.

The authors emphasize that Mengoli's proof did not do this—he was as wary of infinity as the ancient Greeks, from whom he drew direct inspiration. He took care to phrase his proof in finite terms: he applied the above inequality to blocks of 3 terms, then 9 terms, then 27 terms, and argued that by adding enough of these blocks, he could find a partial sum of the harmonic series that must exceed any given number. This is very much in keeping with Archimedes' approach to determining the area of a segment of a parabola, in which he in effect evaluates a Riemann sum by ruling out every possible value for the area except for one.

This engaging article will draw in anyone who has thought about infinite series. One of its most appealing aspects is how it puts the reader in contact with Mengoli's original manuscript—the reader not only sees Mengoli's mathematics, but hears his voice. For instance, he begins with a meditation on Archimedes' determination of areas relating to parabolas. After summarizing Archimedes' argument, Mengoli interjects: "That wonderful theorem!" The reader comes away marveling at how mathematics can be a conversation across centuries.

Response

We are delighted with this honor. It is very encouraging to see others share our excitement about the history of mathematics and our conviction that reflective engagement with the past has a natural place in current mathematical thought. As mathematicians with esoteric interests in history and philosophy, we are very fortunate to have such readers. Some forces in academia would rather push historians to a humanities department and hire another algebraic geometer in their place, but, thanks to the MAA, a more inclusive point of view is alive and well in the mathematical community. In this way, the thriving MAA community makes work such as ours possible through its promotion of a diversity of scholarly approaches to mathematics. We are immensely grateful for this invaluable support.

Biographical Sketches

Jordan Bell is a mathematician and data scientist working in Toronto, Canada. His earlier work as a scholar in the history of mathematics includes a translation of Euler's paper finding the sum of reciprocals of the squares, and an exhaustive review paper on Euler's work on the pentagonal number theorem. He received his MSc in mathematics from the University of Toronto. For the history of mathematics, Jordan's next project is a paper showing that the focus of Book I of Euclid's *Elements* is application of areas (I.44)—not the Pythagorean theorem (I.47)—and to present different medieval Latin proofs that fix a gap in Euclid's proof that has been seldom commented on in modern writings. (Heath does not mention it in his translation, Vitrac does.)

Viktor Blåsjö is an assistant professor at the Mathematical Institute of Utrecht University. He is a historian of mathematics with a special interest in the interplay between technical mathematical content and foundational issues in the early modern period. You can follow him on Twitter @viktorblasjo and listen to his Opinionated History of Mathematics podcast.