The Sine of a Single Degree

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Though the *radian* is the undisputed king of angular measurement in calculus, required as it is for the familiar trigonometric derivative and integration formulas to hold true, it is the *degree* to which most of us are first introduced. The humble degree—one three-hundred-sixtieth part of a complete rotation—is one of the oldest methods used to measure angles. Its use dates to antiquity and it can be found in ancient Babylonian, Greek, Indian, and Mayan mathematical traditions [**1, 4**]. Many theories have been suggested for its ubiquity, ranging from the astronomical (a solar year has approximately 360 days) [**2**, p. 37] to the practical (360 has *lots* of divisors) [**8**, pp. 10–11], but the fact remains that the degree is, if not a mathematically ideal way to measure angles, a particularly *human* way to do so.

In a typical trigonometry class we devote time learning those special angles that admit exact trigonometric ratios involving roots of integers, such as 30° and 45°. A question that might interest students of trigonometry is whether this is also true of the archetypal measure 1°? That is, is its sine expressible as a ratio of some combination of radicals and integers? Can we compute the sine of a single degree exactly?

The answer to this question is "yes," but the path to computing it meanders through classic geometry, polynomial algebra, and complex numbers. The purpose of this paper is to follow this path to value of $\sin 1^\circ$ and to see some beautiful mathematics along the way.

The geometry of sine one degree

Since trigonometry literally means "triangle measurement," a reasonable place to start our investigation is with the geometry of triangles. The familiar right-triangle formulation of the trigonometric functions can be traced back to both ancient Chinese [**13**, pp. 56–71] and Indian [**9**, p. 7] traditions. Given a right triangle with an acute angle θ and a hypotenuse of length 1, the sine of θ is the length of the side opposite the angle. The length of the remaining "adjacent" side is precisely the sine of the *co*mplementary angle, or "co-sine," as is therefore related to the sine by the pair of equations

$$\sin(90^\circ - \theta) = \cos\theta = \sqrt{1 - \sin^2\theta}.$$

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The sine of an angle can also be computed via isosceles triangles. Consider an isosceles triangle with vertex angle 2θ and congruent sides of length 1. Since the altitude of an isosceles triangle bisects both the vertex angle and the base, we have

$$\sin\theta = \frac{1}{2} \times \text{base of this isosceles triangle.} \tag{1}$$

The two most familiar isosceles triangles are the equilateral (60–60–60 degree) triangle and the right-isosceles (45–45–90) triangle, from which the familiar sine formulas $\sin 30^\circ = 1/2$ and $\sin 45^\circ = \sqrt{2}/2$ follow immediately.

Another important triangle, less familiar to us nowadays but was well known to the Greeks, is the *golden triangle*: the isosceles triangle whose base angle is twice its vertex angle [**6**]. We use this golden triangle to compute the sine of 36°.

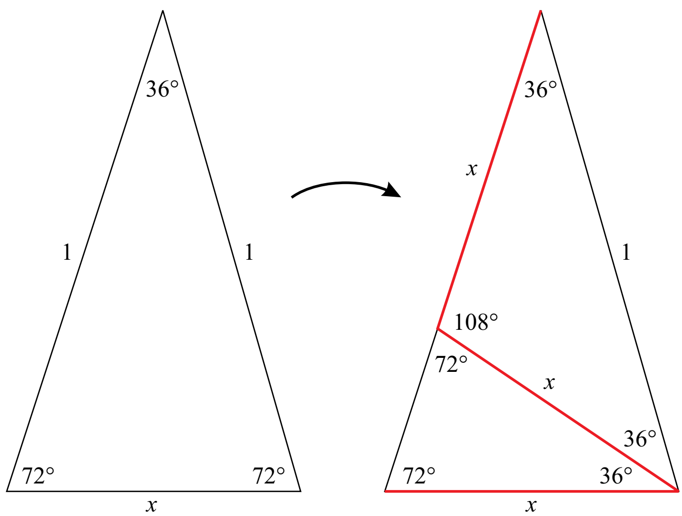

Figure 1. The golden triangle subdivided.

Assume the congruent sides are length 1 and denote the length of the base by x. An angle bisector constructed through one of the base angles divides the golden triangle into the two smaller isosceles triangles depicted in Figure 1: a short, squat 36–36–108 triangle with sides of lengths x, x, and 1; and a tall, skinny 36–72–72 triangle with sides of length x, x, and $1 - x$. This latter triangle is another golden triangle, so it is similar to the larger one. Comparing ratios of side lengths gives

$$\frac{x}{1-x} = \frac{1}{x},$$

from which we conclude $x = (\sqrt{5} - 1)/2$. Greeks recognized this as the *golden mean* and is the reason the triangle earned its name. Applying (1), we conclude

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}. \tag{2}$$

Can we deduce the sine of yet smaller angles? If we can construct two angles $\alpha > \beta$, then it is also possible to construct their difference $\alpha - \beta$. Moreover, if we know the exact values of the trigonometric functions at α and β, then it is possible to determine the exact values at $\alpha - \beta$, namely

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

A clever geometric proof of these formulae can be found in [**12**, pp. 46–47], which we show in Figure 2; proofs *with* words can be found in any trigonometry textbook.

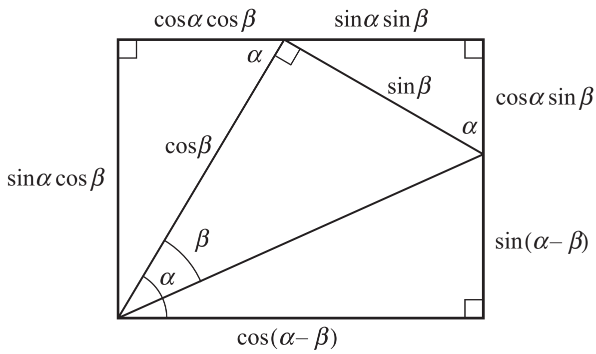

Figure 2. Verifying the angle-subtraction formulas.

Using this process, how small an angle can we obtain? Given that we know the sine and cosine at 45° and 30°, we can compute

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}, \tag{3}$$
$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}.$$

Combining these two values with the trigonometric values at $\alpha = 18^\circ$ from (2), we can work down to 3°:

$$\sin 3^\circ = \frac{\sqrt{2}\left(\sqrt{3}+1\right)\left(\sqrt{5}-1\right) - 2\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}}{16}. \tag{4}$$

This is just a stone's throw away from $\sin 1^\circ$, but sadly, that is as close as we are going get using only the Greek geometer's tools of straightedge and compass. None of the Greek geometers—or indeed *any* of the geometers to come after—could find a straightedge-and-compass construction that would yield 1°. It would not be until centuries later that such a construction was shown to be impossible, a consequence of the powerful algebraic theorems devised by mathematicians in the 18th and 19th centuries [**14**]. Perhaps the most relevant result to us is the following: An angle of integer degree measure can be trisected if and only if it is a multiple of 9°. Since this is not true of 3°, we cannot construct 1°. Interested readers can find a wonderful survey of the algebraic results regarding the constructibility of angles in the Pólya Award winning article [**7**].

The algebra of sine one degree

While the tools of algebra are enough to scuttle the constructibility of 1°, they can (fortuitously) be used to salvage its sine. It is possible to algebraically manipulate trigonometric identities to *solve* for $\sin 1^\circ$ as the solution of an equation rather than as the length of a side of a very specific triangle.

For example, if we repeatedly apply the angle-addition formulas for sine and cosine (which the interested reader is invited to obtain by modifying the labels on Figure 2), we find

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta, \tag{5}$$

which algebraically relates the sine of an angle to the sine of thrice the angle. Setting $\theta = 15^\circ$ in the "triple-angle formula" (5) gives

$$\sin 45^\circ = 3\sin 15^\circ - 4\sin^3 15^\circ = \frac{\left(\sqrt{3}+1\right)\sqrt{2-\sqrt{3}}}{2}, \tag{6}$$

using the exact value of $\sin 15^\circ$ from (3). After a moment's pause (or panic?), we see that (6) is equal to a more familiar value $\sin 45^\circ = \sqrt{2}/2$ by squaring both sides.

Since we know the exact value of sine at 3°, setting $\theta = 1^\circ$ in (5) gives

$$\sin 3^\circ = 3\sin 1^\circ - 4\sin^3 1^\circ,$$

showing that $x = \sin 1^\circ$ is a solution to the polynomial equation $\sin 3^\circ = 3x - 4x^3$. Isolating the x^3 term and using the exact expression for $\sin 3^\circ$ from (4), this becomes

$$x^3 = \frac{3}{4}x - \frac{\sqrt{2}\,(\sqrt{3}+1)(\sqrt{5}-1) - 2\,(\sqrt{3}-1)\sqrt{5+\sqrt{5}}}{64}. \tag{7}$$

Figure 3 shows a plot of the curves $y = x^3$ and $y = (3x - \sin 3^\circ)/4$, which clearly intersect three times, so the solution closest to the origin must be the exact value of $\sin 1^\circ$, but how do we find it?

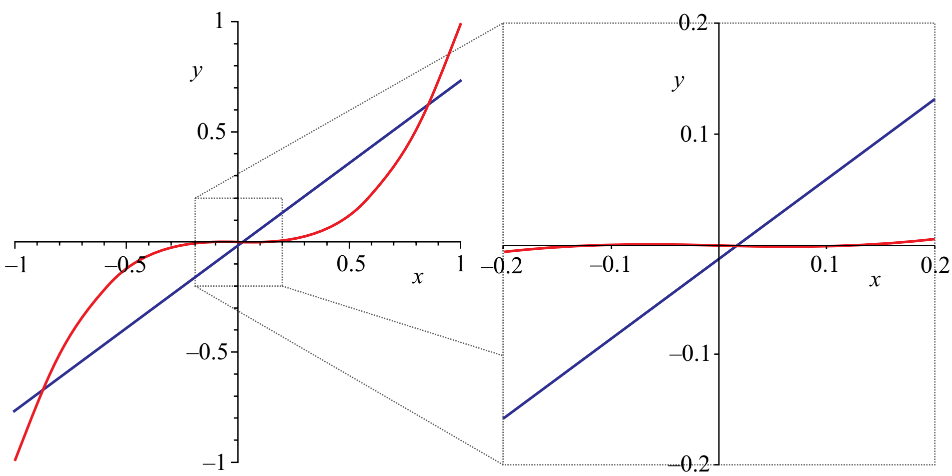

Figure 3. Solutions to the $\sin 1^\circ$ cubic equation.

The solution to this equation, or rather, the more general "depressed cubic" equation

$$x^3 = Ax + B, \tag{8}$$

is one of the great triumphs of algebra. Discovered first by Italian mathematician Niccolo Tartaglia and communicated to the world by Girolamo Cardano in the mid-16th century, the basic idea behind the solution is to assume that the value x is a difference of two quantities, say $x = t - u$. Using algebraic manipulations, it follows that

$$x^3 = (t-u)^3 = -3tu(t-u) + (t^3 - u^3) = -3tu\,x + (t^3 - u^3). \tag{9}$$

What makes this formula such a wonder is that Tartaglia and Cardano derived it entirely geometrically as a statement about the volume of a large cube with a smaller cube removed from one corner! A wonderful description of its discovery, and the feud that eventually resulted between the two mathematicians, can be found in [**5**, pp. 133–154]. Comparing (8) and (9), if we can find values of t and u that solve the pair of equations

$$A = -3tu, \quad B = t^3 - u^3, \tag{10}$$

then $x = t - u$ will solve the original depressed cubic (8); so let us solve (10) for t and u.

Solving both equations simultaneously for u^3 yields

$$-\frac{A^3}{27t^3} = t^3 - B,$$

which can be rewritten as

$$(t^3)^2 - B\,t^3 + \frac{A}{27} = 0.$$

This is a quadratic in t^3, so applying the quadratic formula and taking the positive radical (as Cardano would have) gives

$$t^3 = \frac{B}{2} + \sqrt{\frac{B^2}{4} - \frac{A^3}{27}}.$$

Taking the cube root of this yields t, which we can then use to solve for u to obtain

$$x = t - u = \sqrt[3]{\frac{B}{2} + \sqrt{\frac{B^2}{4} - \frac{A^3}{27}}} - \sqrt[3]{-\frac{B}{2} + \sqrt{\frac{B^2}{4} - \frac{A^3}{27}}},$$

as a solution of the depressed cubic.

Now we apply this to the problem at hand. If we set $A = 3/4$ and $B = -(\sin 3^\circ)/4$, then we recover (7). Using the Tartaglia–Cardano technique, it follows that one of the solutions must be

$$x = \sqrt[3]{-\frac{\sin 3^\circ}{8} + \sqrt{\frac{\sin^2 3^\circ}{64} - \frac{1}{64}}} - \sqrt[3]{\frac{\sin 3^\circ}{8} + \sqrt{\frac{\sin^2 3^\circ}{64} - \frac{1}{64}}}. \tag{11}$$

Before we declare victory and call this value "sin 1°," recall that (7) has *three* real solutions, so this x might be one of the other values. Unfortunately for us, numerically evaluating the expression in (11) on a calculator gives

$$x \approx -0.009 + 0.015\,i$$

where i denotes the so-called imaginary number, i.e., $i = \sqrt{-1}$. That is, the x given by (11) is *not* even a real number, much less one of the solutions to our depressed cubic equation, and much *much* less our elusive sin 1°!

Where did we go wrong? With a little algebra, we can rewrite (11) as

$$\begin{aligned} x &= \frac{\sqrt[3]{-\sin 3^\circ + \sqrt{\sin^2 3^\circ - 1}} - \sqrt[3]{\sin 3^\circ + \sqrt{\sin^2 3^\circ - 1}}}{2} \\ &= \frac{\sqrt[3]{-\sin 3^\circ + i\ \cos 3^\circ} - \sqrt[3]{\sin 3^\circ + i\ \cos 3^\circ}}{2} \end{aligned} \tag{12}$$

and the involvement of the imaginary number becomes jarringly apparent.

The Tartaglia–Cardano method occasionally involves extracting the cube root of complex (i.e., imaginary) numbers. Although both mathematicians knew this, neither they nor their algebraic successors knew how to make sense of them. No algebraist of their time knew how to systematically compute the cube root of an imaginary number. No algebraist even knew what a cube root of an imaginary number *meant*!

The complex arithmetic of sine one degree

The problem of the imaginary cube roots would not be resolved until later in the 18th century, when mathematicians finally looked past the bizarre arithmetic of imaginary numbers and found very real geometry working behind the scenes. A wonderful history of the de-mystification of imaginary numbers can be found in [**10**].

The essential idea—indeed, the typical starting point for a modern discussion of complex arithmetic such as [**3**] or [**11**]—is to equate a complex number such as $z = 2 + 3i$ with a position vector in the plane, here $\vec{z} = \langle 2, 3\rangle$. In particular, the infamous "imaginary" number is nothing more than the (second) standard basis vector $\langle 0, 1\rangle$.

Not only does this identification put the complex numbers squarely in the "real" universe, but it provides some geometric insight into the arithmetic of complex numbers. For example, addition of complex numbers is the same as the usual addition of vectors save for a change of notation:

$$(2 + 3i) + (4 + i) = 6 + 4i \quad \Longleftrightarrow \langle 2, 3\rangle + \langle 4, 1\rangle = \langle 6, 4\rangle.$$

Of course, it was not the additive properties of complex numbers that befuddled mathematicians; it was their multiplicative habit of squaring to negative numbers. Remarkably, viewing complex numbers as vectors gives a simple geometric description of complex multiplication as well. To see it, imagine for a moment we have a complex number z, which we represent as a vector. If z has length r and direction angle θ (measured from the positive x-axis), it can be written as

$$z = \langle r\cos\theta, r\sin\theta\rangle = r\cos\theta + i\,r\sin\theta$$

using (modern) trigonometry. Consider a second complex number (or plane vector, whichever you prefer) $w = s\cos\varphi + i\,s\sin\varphi$ and compute the product zw by expanding the terms and recalling that $i^2 = -1$:

$$\begin{aligned} zw &= (r\cos\theta + i\,r\sin\theta)(r\cos\varphi + i\,r\sin\theta) \\ &= rs(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i\,rs(\sin\theta\cos\varphi + \cos\theta\sin\varphi). \end{aligned}$$

That may not look like an improvement, but the expressions in parentheses are precisely the angle addition formulas for sine and cosine! That means

$$zw = rs\cos(\theta + \varphi) + i\,rs\sin(\theta + \varphi),$$

that is, zw is a vector of length rs that makes a $\theta + \varphi$ angle with the positive x-axis. Said differently, to multiply two plane vectors as complex numbers, we multiply their lengths and add their direction angles.

This simple geometric description gives a nice explanation of why the product of two negative real numbers is a positive one: Every negative number makes an angle of 180° with the positive x-axis, so when they are multiplied together their angles are added to obtain 360°, yielding a vector pointing in the direction of the positive x-axis. (Can you now *see* why it also shows that $i^2 = -1$?)

Perhaps more germane to our problem, it also explains how complex cube roots work. If z is a complex number with vector magnitude r and direction angle θ, then z^3 is a vector with magnitude r^3 and direction angle 3θ. Hence, to find a complex number that cubes to $8i$, we need only a vector whose length cubes to 8 (i.e., the length of $8i$) and whose angle triples to be coterminal with 90° (i.e., the angle of $8i$). Obviously, a vector of length 2 and angle 30° should work, so that

$$2\cos 30^\circ + i\,2\sin 30^\circ = \sqrt{3} + i$$

is a cube root of $8i$. The reader is encouraged to cube this by algebraic expansion to see that it does indeed equal $8i$.

However, unlike the case for real numbers, this is not the only cube root. For example, a vector of length 2 and angle 150° would cube to a vector of length 8 and angle 450° which is coterminal to 90°. Thus,

$$2\cos 150^\circ + i\,2\sin 150^\circ = -\sqrt{3} + i$$

is another cube root.

Are there other cube roots as well? If so, they would need an angle θ that, when tripled, is coterminal with 90°. If we solve equation $3\theta = 90^\circ + 360^\circ n$, we find $\theta = 30^\circ + 120^\circ n$. For $n = 0$, we recover our first cube root; for $n = 1$, we recover our second one. If $n = 3$, we find $\theta = 270^\circ$, whence

$$2\cos 270^\circ + i\,2\sin 270^\circ = -2i$$

is a third cube root (one we probably could have guessed at on our own). Continuing on, for $n = 3$, we again recover $\sqrt{3} + i$ again, therefore we can conclude that there are exactly *three* cube roots to $8i$, each of which is rotated exactly 120° from the other two.

This conclusion holds in general: Every nonzero complex number z has *three* distinct cube roots. This collection of three roots is denoted $z^{1/3}$ and this is called the *algebraic cube root* of z. Of the three, the root with the minimum direction angle (in absolute value) is called the *principal* cube root and is denoted $\sqrt[3]{z}$. In the example above, we would have

$$(8i)^{1/3} = \left\{\sqrt{3}+i, -\sqrt{3}+i, -2i\right\} \quad \text{whereas} \quad \sqrt[3]{8i} = \sqrt{3}+i.$$

That complex numbers have multiple cube roots explains precisely the problem we faced in our algebraic determination of $\sin 1^\circ$. When a modern calculator computes a complex cube root, it determines only the *principal* root. This is why the algebraic solution in (11) failed—the cube roots we demand of the solution are not necessarily the principal cube roots. If we return to our discussion of the solution to the depressed cubic equation (7), we realize that instead of (12), a more accurate description of the Tartaglia–Cardano solution is

$$x \in \frac{(-\sin 3^\circ + i\ \cos 3^\circ)^{1/3} - (\sin 3^\circ + i\ \cos 3^\circ)^{1/3}}{2}. \tag{13}$$

Unfortunately, this no longer defines a single value, but rather *nine* different values, three for each cube root. How do we sift among them to find the value $\sin 1^\circ$?

One approach is to graph the nine complex numbers described in (13). For convenience, define

$$z = -\sin 3^\circ + i \cos 3^\circ \quad \text{and} \quad w = \sin 3^\circ + i\ \cos 3^\circ$$

so that $x \in (z^{1/3} - w^{1/3})/2$. By the co-function property, $z = -\cos 87^\circ + i \sin 87^\circ$ also, a unit vector that makes an 87° angle with the *negative* x-axis. Thus, z makes a 93° angle with the positive x-axis so its three cube roots will be unit vectors that make angles of 31°, 151°, and 271°. That is,

$$z^{1/3} = \left\{ \begin{array}{c} \cos 31^\circ + i\ \sin 31^\circ, \\ \cos 151^\circ + i\ \sin 151^\circ, \\ \cos 271^\circ + i \sin 271^\circ \end{array} \right\}.$$

The relationship is shown in Figure 4 left, with z appearing in red/bold. A similar argument shows

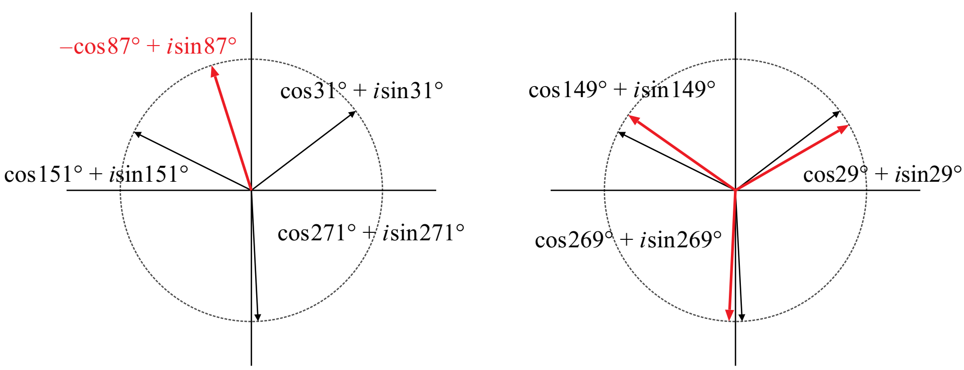

Figure 4. The cube roots of z and w.

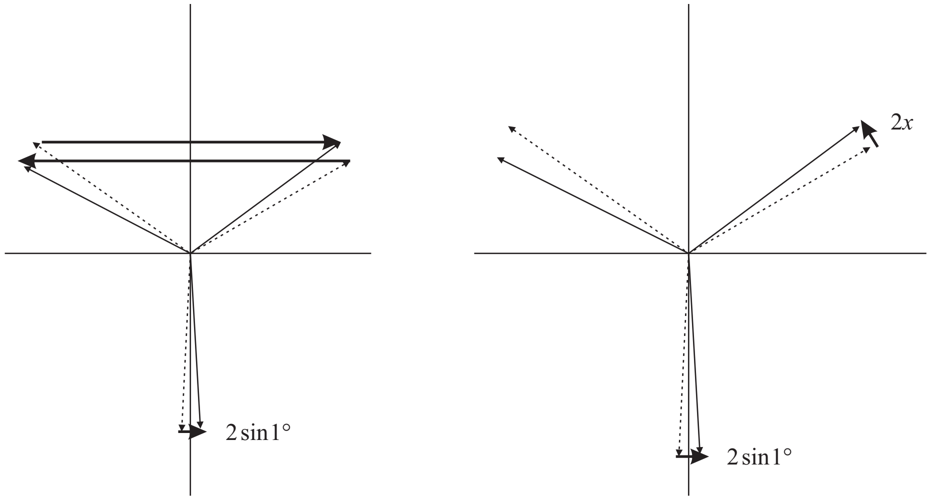

Figure 5. Rotating the erroneous solution to get sin 1°.

$$w^{1/3} = \left\{ \begin{array}{c} \cos 29^\circ + i\ \sin 29^\circ, \\ \cos 149^\circ + i\ \sin 149^\circ, \\ \cos 269^\circ + i \sin 269^\circ \end{array} \right\}.$$

These vectors appear in Figure 4 right in red/bold, together with the three cube roots of z for comparison.

On the one hand, the expression $z^{1/3} - w^{1/3}$ can represent any of the vectors formed as a vector difference of one of the three vectors representing $z^{1/3}$ with one of the vectors representing $w^{1/3}$ or, equivalently, any vector connecting a $w^{1/3}$ vector with a $z^{1/3}$ one. Of the nine such arrows, exactly three of them are horizontal, shown in Figure 5 left. These vectors represent three real numbers, two positive and one negative, consistent with the graphical information we found in Figure 3. In fact, the bottom-most real vector, formed by connecting the two cube roots located exactly 1° on either side of the negative y-axis, is the base of an isosceles triangle with equal sides length 1, thus the base length is $2 \sin 1^\circ$ by (1).

On the other hand, the expression $\sqrt[3]{z} - \sqrt[3]{w}$, which equals twice our original erroneous Tartaglia–Cardano solution x from (11), represents exactly one vector: the difference between the two *principal* cube roots. As these two roots are both located in the first quadrant, it follows that $2x$ is the small vector connecting the tip of $\sqrt[3]{w}$ to the tip of $\sqrt[3]{z}$, illustrated in Figure 5 right. Since each cube root of z (or w) is a rotated copy of the principal cube root, we need only rotate $2x$ clockwise 120° to make it horizontal, and hence match $2 \sin 1^\circ$ exactly.

Using the geometry of complex multiplication, this means we need only multiply the incorrect x in (12) by $\cos(-120^\circ) + i\ \sin(-120^\circ)$ to obtain the desired value:

$$\begin{aligned} \sin 1^\circ &= \left(-\frac{1}{2} - i\,\frac{\sqrt{3}}{2} \right) \frac{x}{2} \\ &= \frac{1 + i\sqrt{3}}{4} \left(\sqrt[3]{\sin 3^\circ + i\ \cos 3^\circ} - \sqrt[3]{-\sin 3^\circ + i\ \cos 3^\circ} \right). \end{aligned}$$

Using the explicit formula for $\sin 3^\circ$ given in (4) and the Pythagorean identity, we can finally express the exact value of $\sin 1^\circ$. Replacing i by $\sqrt{-1}$, we can then write $\sin 1^\circ$ in terms of radicals and integers as

$$\frac{1+\sqrt{-3}}{8}\left(\sqrt[3]{\begin{aligned}&(\sqrt{5}-1)\sqrt{2+\sqrt{3}}-\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{2-\sqrt{3}}\\&\quad+\sqrt{\left((\sqrt{5}-1)\sqrt{2+\sqrt{3}}-\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{2-\sqrt{3}}\right)^2-64}\end{aligned}}\right.$$

$$\left.-\sqrt[3]{\begin{aligned}&(1-\sqrt{5})\sqrt{2+\sqrt{3}}+\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{2-\sqrt{3}}\\&\quad+\sqrt{\left((\sqrt{5}-1)\sqrt{2+\sqrt{3}}-\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{2-\sqrt{3}}\right)^2-64.}\end{aligned}}\right) \tag{14}$$

Epilogue

The expression (14) is an eyeful, but the complex approach can suggest other radical expressions for $\sin 1^\circ$. It is clear that

$$(\cos 1^\circ + i\ \sin 1^\circ) - (\cos 1^\circ - i\ \sin 1^\circ) = 2i \sin 1^\circ.$$

However, using the properties of complex roots, we can also conclude that

$$\cos 1^\circ + i\ \sin 1^\circ = \sqrt[n]{\cos n^\circ + i \sin n^\circ}$$

for any n with $1 \le n \le 90$ where we are using the principal nth root. Combining these two observations, it follows that

$$\sin 1^\circ = \frac{1}{2i}\left(\sqrt[n]{\cos n^\circ + i\ \sin n^\circ} - \sqrt[n]{\cos n^\circ - i\ \sin n^\circ}\right)$$

for any n with $1 \le n \le 90$. This provides an unusual way to compute $\sin 1^\circ$ in terms of radicals of trigonometric functions at angles we may already know.

For example, $n = 3$ gives (14) in slightly rearranged form and $n = 30$ yields

$$\sin 1^\circ = \frac{1}{2\sqrt{-1}}\left(\sqrt[30]{\frac{\sqrt{3}+\sqrt{-1}}{2}} - \sqrt[30]{\frac{\sqrt{3}-\sqrt{-1}}{2}}\right),$$

which is at least prettier to look at. Using $n = 45$ gives the similarly concise

$$\sin 1^\circ = \frac{1}{2\sqrt{-1}}\left(\sqrt[45]{\frac{1+\sqrt{-1}}{\sqrt{2}}} - \sqrt[45]{\frac{1-\sqrt{-1}}{\sqrt{2}}}\right).$$

Perhaps the most exotic description for $\sin 1^\circ$ comes from the extreme case $n = 90$, for which the cosine term vanishes and the sine term becomes 1:

$$\sin 1^\circ = \frac{\sqrt[90]{\sqrt{-1}} - \sqrt[90]{-\sqrt{-1}}}{2\sqrt{-1}},$$

which, in addition to being delightfully bizarre and strangely beautiful, has the added bonus of being true.

Acknowledgment. The author would like to thank the professors and students of the South Dakota School of Mines and Technology for both their comments and suggestions to improve the paper and their patience for listening to him present it over and over again.

Summary. Ostensibly a derivation of an algebraically exact formula for the value of the sine of 1 degree, we present this calculation as a "historical romp" looking at the problem through the tools of geometry, then algebra, and finally complex analysis. Each one of these approaches gets the reader nearer to the correct value, but also serves to frame a vignette of surprising or beautiful mathematics.

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