Biangular Coordinates Redux
Discovering a New Kind of Geometry

Michael Naylor and Brian Winkel

Michael Naylor (mnaylor@cc.wwu.edu) teaches mathematics and mathematics education at Western Washington University in Bellingham, Washington. He studies a broad spectrum of mathematical ideas including geometry, pedagogy, and mathematical art.

Brian Winkel (Brian.Winkel@usma.edu) teaches and mentors mathematics, students, and faculty at the United States Military Academy. He is the founder and Editor-in-Chief of *PRIMUS – Problems, Resources, and Issues in Mathematics Undergraduate Studies* and is founder and Editor Emeritus of *Cryptologia*.

Coordinate systems are methods for describing the location of points in space. The two most popular systems in two dimensions are Cartesian (or rectangular) coordinates and polar coordinates; the former describes points using two distances (x and y) and the latter relies on one distance and one angle (r and θ). The natural complement of these systems is one in which location (point P) is determined using two angles, ϕ and θ. This system is the *biangular coordinate system* (see Figure 1).

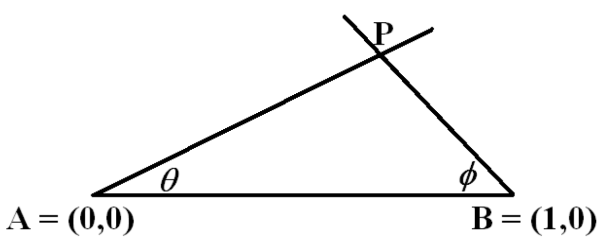


Figure 1: Biangular coordinates.

We consider two points $A = (0,0)$ and $B = (1,0)$ in the xy-plane. We call these points *poles* and the x-axis through A and B *the polar axis.* $(\theta, \phi)_b$ signifies *biangular coordinates* of a point P formed by the intersection of two rays through P, one from A at angle θ (measured counterclockwise) from the polar axis AB and the other from B at angle ϕ (measured clockwise) from the polar axis AB.

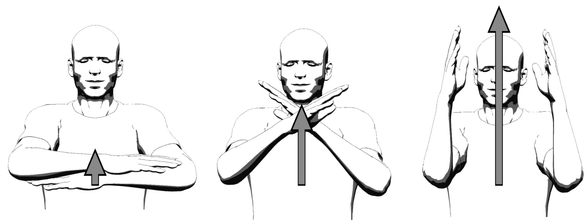


Figure 2: Hand waving for the biangular relation $\phi = \theta$.

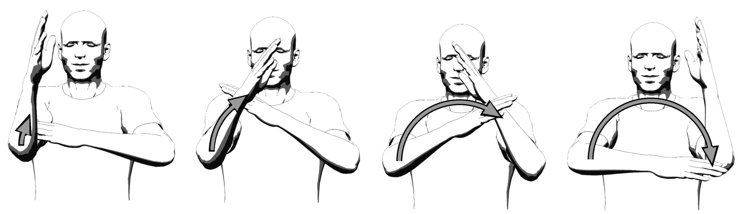


Figure 3: Hand waving for the biangular relation $\phi = \frac{\pi}{2} - \theta$.

Try this yourself by holding your arms vertically upward in front of you and allowing your elbows to be the poles (see Figure 2). Keeping your elbows in place, move your forearms in the plane parallel to the front of your body. Your forearms represent rays from the *poles*, each making an angle from oppositely directed horizontal rays, one from point A and one from point B, and the intersection of these rays is the biangular coordinate or point, P, described by these angles — the left arm forms an angle, θ, with the horizontal from A, while the right arm forms an angle, ϕ, with the horizontal from B.

Hold your forearms horizontally, i.e. $\theta = 0$ and $\phi = 0$, (see Figure 2) and then increase the angles at the same rate. What curve is described by the loci of intersections of your arms (extended if necessary)? Next, start with the left forearm horizontal, $\theta = 0$, and the right forearm vertical, $\phi = \frac{\pi}{2}$ (see Figure 3). Decrease ϕ at the same rate that you increase θ. What curve does the loci of intersections describe this time? Do you "feel" or "see" $\phi = \frac{\pi}{2} - \theta$?

The study of biangular coordinates has an interesting history, offers students a rich environment in which to study trigonometry, geometry, and functions, permits students to ask "what if" questions and discover patterns and relations, and leads to many surprising connections and insights.

We have found that the study of biangular coordinates has an interesting history, is enjoyable to the senses, offers students a rich environment in "what if" questions and discover patterns and relations on their own, and leads to many surprising connections and insights. While those who studied biangular coordinates in earlier times (mostly 19^{th} century England) were inclined to "live" in a biangular plane, this survey instead takes a biangular view of rectangular coordinates.

DEFINITION AND NOTATION

To begin, we need to convert biangular coordinates $(\theta,\phi)_b$ to rectangular coordinates. Figure 4 demonstrates how to find the rectangular coordinates of the point $P = (\theta,\phi)_b$: $x = x(\theta,\phi)$ is the length of the line segment $\overline{AQ}$, $1 - x(\theta,\phi)$ is the length of the line segment $\overline{QB}$, and $y = y(\theta,\phi)$ is the length of the line segment $\overline{QP}$. The point P is at the intersection of the rays from A and B.

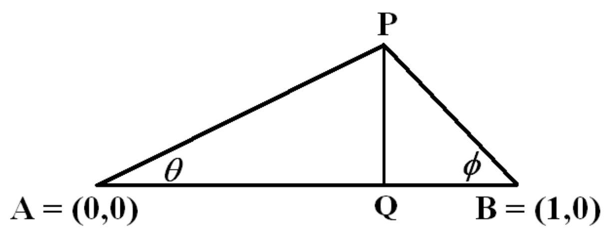


Figure 4: Determining the rectangular coordinates of P, given biangular coordinates.

If we have a relationship between ϕ and θ, say $\phi = f(\theta) = \theta + 2$ or $F(\theta,\phi) = \cot(\theta) + \cot(\phi) - 1 = 0$, we call such a function or relation a *biangular relation*, and the resulting plot of all points P determined by the above procedure, the *biangular plot* of the biangular relation.

When studying biangular coordinates and biangular relations with students we leave the determination of the rectangular coordinates of $P = (\theta,\phi)_b$ as an introductory activity. Here is the way they usually attack the problem. They drop a perpendicular from P to a point Q on the polar axis $\overline{AB}$. In the right triangle APQ, $\tan(\theta) = \frac{|\overline{PQ}|}{|\overline{AQ}|}$, and in right triangle BPQ, $\tan(\phi) = \frac{|\overline{PQ}|}{|\overline{BQ}|}$. Now, solving these equations for $|\overline{AQ}|$ and $|\overline{BQ}|$, then simplifying and using the trigonometric identity for $\sin(\theta+\phi)$, the rectangular representation for the point $P = (\theta,\phi)_b$ is:

$$P = \left(\frac{\cos(\theta)\sin(\phi)}{\sin(\theta+\phi)},\ \frac{\sin(\theta)\sin(\phi)}{\sin(\theta+\phi)} \right) \tag{1}$$

To plot a specific biangular relation, say $\phi = f(\theta)$ or $F(\theta,\phi) = 0$, we use a parametric plot routine for a biangular function, and something like the `ContourPlot` command in *Mathematica*, or an implicit plotting routine for a biangular relation which is not a function. Once there is graphical representation, there follows a euphoric period of play for the biangular novice in which exploration takes place and a wide variety of biangular relations are plotted. Patterns begin to emerge.

HISTORICAL BACKGROUND

Newton, immediately after computing derivatives in rectangular coordinates, turns to no less than eight different coordinate systems and computes derivatives in each.[**9,** p. 52] He calls these systems "manners."

> However it may not be foreign from the purpose, if I also shew how the problem may be perform'd, when the curves are refer'd to right lines, after any other manner whatever;

> so that having the choice of several methods, the easiest and most simple may always be used.[**9,** p. 51]

He does not use a biangular coordinate system. However, we surmise that his eight coordinate systems are but a sample. He makes it clear that his interest in other coordinate systems is for ease in computing the tangent to the curve, but he also says other coordinate systems may be appropriate for their convenience in studying other situations "as occasion shall require."[**9,** p. 55]

The first published instance of biangular plots occurs in 1803 in the *Géométrie de position* by Lazare Nicholas Marguerite Carnot.[**3,** p. 466] Carnot devoted considerable attention to biangular plots in his chapter *De la détermination d'un point dans l'espace, et du changement de ses coordinates*. He studied the biangular relations $\theta + \phi = m$, for constant m, and verified that these form circles while showing that the relations $\theta - \phi = m$ produce hyperbolae. He derived Equation (1).

During the last half of the 19$^{\text{th}}$ century in England a number of coordinate systems were introduced to study conics. In William Walton's 1868 paper [**10**] biangular coordinates are presented. The paper begins with the following introduction:

> Of the various classes of curves in Algebraic Geometry, some are best represented by bilinear, some by polar coordinates. There exist however numberless curves, many of the properties of which it would be inconvenient to investigate by the aid of either of these systems of coordinates.[**10,** p. 47]

Walton then proceeds to offer a number of results concerning the biangular form of a straight line, parallel and perpendicular lines, and tangent lines to biangular coordinate curves.

Robert Williams Genese, in 1882, offers an application of biangular coordinates for the following problem:

> Given the base AB of a triangle, and the locus $f(\alpha, \beta) = 0$ of the vertex P, to find the locus of the centres inscribed and escribed of the triangle APB.[**6,** p. 152]

The most extensive papers on biangular coordinates are by Genese [**7**] and T. Biggin [**2**]. Genese's paper of 21 pages was simply entitled, "Biangular Coordinates" [**7**].

In his 1888 paper [**7**] Genese gives the formal definition of biangular coordinates and then proceeds to study straight lines and conics in biangular coordinates, with short proofs that take advantage of biangular coordinates for the propositions that the perpendicular bisectors of a triangle meet in a point as do the angle bisectors. Biggin [**2**], in his 1891 paper, discusses how to convert between rectangular and biangular coordinate systems. Like Genese [**6**] in his 1882 paper, Biggin uses calculus and biangular coordinates while he studies such topics as curvature.

In 1910, in the first work in an American journal [**12**] on this subject, G. B. M. Zerr begins with the following:

> It is the purpose of this discussion to set forth a very elementary exposition of a most interesting subject rather than anything new, with the hope that it may be further developed in future issues. In what follows the notation of Professor Genese is used.[**12,** p. 34]

There is no published paper on the subject after Zerr's paper in 1910. The most recent available work on biangular coordinates is the 1950 Master's thesis of Howard W. Baeumler [**1**].

THE FAMILY OF FUNCTIONS: $\phi = m\theta + k$

Consider the family of linear functions $\phi = m\theta + b$ for all values of m and b. In rectangular coordinates we devote a great deal of time to linear functions as we consider slope, intercept, approximating a curve with a line, etc. Thus it is quite natural for students to ask what the biangular relations $\phi = m\theta + b$ yield. Students enjoy exploring this family and produce a wonderful variety of curves with much visual appeal.

Line: $\phi = \theta$. We depicted the "feel" of this relation in Figure 2. The plot of this biangular relation is the vertical line through the midpoint of the line connecting the poles A and B. When $\phi = -\theta$, the rays are parallel and therefore do not intersect in a single point. Every point on the line containing the polar axis, however, satisfies the relation $\phi = -\theta$.

Circles: $\phi = -\theta + k$. The biangular relations $\phi = -\theta + k$ produce circles which all go through the poles A and B. From these plots and a bit of geometry we can see that the circle has radius $r = \left|\frac{1}{2\sin(k)}\right| = \left|\frac{\csc(k)}{2}\right|$ and center $C = \left(\frac{1}{2}, -\frac{\cot(k)}{2}\right) = \left(k - \frac{\pi}{2}, k - \frac{\pi}{2}\right)_b$. These circles all have the line segment $\overline{AB}$ as a secant and when $k = \frac{\pi}{2}$, $\overline{AB}$ is the diameter of the circle. When $0 < k < \frac{\pi}{2}$ the center of the circle, $\left(\frac{1}{2}, -\frac{\cot(k)}{2}\right)$, lies below the polar axis and when $\frac{\pi}{2} < k < \pi$ the center of the circle, $\left(\frac{1}{2}, -\frac{\cot(k)}{2}\right)$, lies above the polar axis. The circle is smallest (and symmetric about the x-axis) when $k = \frac{\pi}{2}$ and largest (infinite radius) when $k = 0$ or $k = \pi$. The latter statements are true for $k = \frac{\pi}{2} \pm n\pi$ and $k = \pm n\pi$ for $n = 0, 1, 2, \ldots$, respectively. See Figure 5.

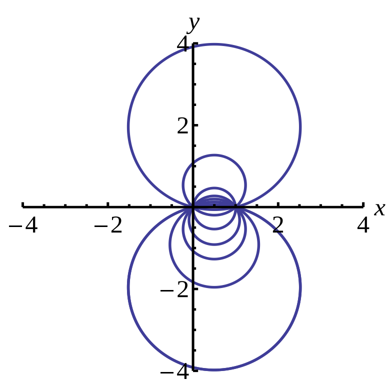


Figure 5: Members of the family $\phi = -\theta + k$ for various values of k.

To calculate the center of this circle, let P be the point on the plot for the biangular relation $\phi = k - \theta$ when $\theta = \frac{\pi}{2}$, i.e. $P = \left(\frac{\pi}{2}, k - \frac{\pi}{2}\right)_b$. See Figure 6.

Since $\angle BAP$ is a right angle then all three points A, B, and P, are on a circle where $\overline{BP}$ is the diameter. Similarly, let Q be the point on the plot when $\phi = \frac{\pi}{2}$, i.e. $Q = \left(k - \frac{\pi}{2}, \frac{\pi}{2}\right)_b$ and here $\overline{AQ}$ is

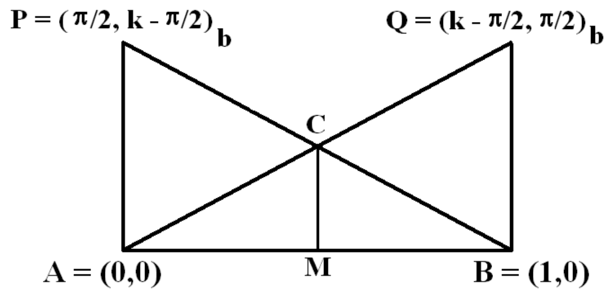


Figure 6: Computing the center and radius of the biangular plot for $\phi = k - \theta$.

another diameter of the circle. The center of this circle lies at the intersection of these two diameters, namely at point $C = \left(k - \frac{\pi}{2}, k - \frac{\pi}{2}\right)_b$. The length of the diameter, d, of this circle is $|\overline{AQ}|$. We can compute this using the cosine of $\angle QAB$, for $\cos\left(k - \frac{\pi}{2}\right) = \frac{|\overline{AB}|}{|\overline{AQ}|} = \frac{1}{d}$. Now, $\sin(k) = \cos\left(k - \frac{\pi}{2}\right) = \frac{1}{d}$ and thus $d = \frac{1}{\sin(k)} = \csc(k)$. Hence, the radius of this circle is $r = \frac{1}{2}d = \frac{\csc(k)}{2}$.

Hyperbola: $\phi = \theta + k$. The two rays from the poles $A = (0, 0)$ and $B = (1, 0)$ will be parallel, at times producing asymptotic behavior. In Figure 7 we trace the hyperbolic plot of the biangular relation $\phi = \theta + \frac{\pi}{2}$. On the plot $\theta = \frac{\pi}{4}$ indicates an asymptote when $\theta = \frac{\pi}{4}$ for here $\phi = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$, so the rays formed by θ and ϕ emanating from poles $A = (0, 0)$ and $B = (1, 0)$ respectively are parallel. The plot "breaks" at $\theta = \frac{\pi}{4}$ and continues in the lower left quadrant, moving toward the point where $\theta = 1.2$ and so on. As θ increases from $\theta = 0$ to $\theta > \frac{\pi}{2}$ the plot breaks again at $\theta = \frac{3\pi}{4}$, giving another asymptote, and then jumps to the lower right quadrant, finally returning again to the starting point at rectangular pole $B = (1, 0)$ when $\theta = \pi$.

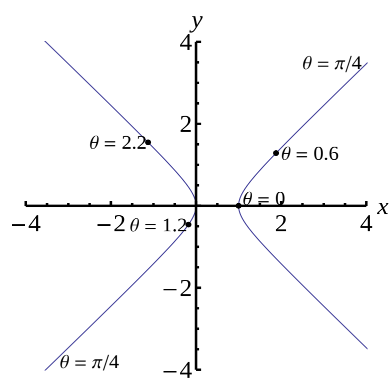


Figure 7: The plot of the biangular relation $\phi = \frac{\pi}{2} + \theta$ with asymptotes.

There are two ways asymptotes can occur—the respective rays from the poles are either in the same direction or in opposite direction. In the case of the same direction, from the definition of the angles θ and ϕ in Figure 1, we see that we shall have asymptotes when θ and ϕ are supplementary, i.e. $\theta + \phi = \pi$. When $\phi = \frac{\pi}{2} + \theta$, this means we have two constraints from which we see $\theta = \frac{\pi}{4}$ and $\phi = \frac{3\pi}{4}$.

We can compute the equation, $Y = mX + b$, of the asymptote as the limit of the tangent lines to the biangular plot as $\theta \to \frac{\pi}{4}$, namely

$$Y = \lim_{\theta \to \frac{\pi}{4}} \left[y(\theta) + \frac{y'(\theta)}{x'(\theta)}(X - x(\theta)) \right] \tag{2}$$

where $\frac{y'(\theta)}{x'(\theta)} = \frac{dy}{dx}$ is the slope of the tangent to the biangular plot in parametric form. We obtain, $Y = X - \frac{1}{2}$ or in our xy-coordinate system $y = x - \frac{1}{2}$. In a similar manner we compute the equation of the other asymptote to be $y = -x + \frac{1}{2}$. We find the intersection of the two asymptotes is $(\frac{1}{2}, 0)$, not surprising, given our plot in Figure 7.

Now we consider asymptotics for the general biangular relation $\phi = \theta + k$. From a similar analysis we can see the two values of θ which give asymptotes are $\theta = \frac{1}{2}(-k + \pi)$ and $\theta = -\frac{k}{2}$. These yield asymptotes $y = \cot\left(\frac{k}{2}\right) x - \frac{1}{2}\cot\left(\frac{k}{2}\right)$ and $y = -\tan\left(\frac{k}{2}\right) x + \frac{1}{2}\tan\left(\frac{k}{2}\right)$ respectively and the intersection for all k values is always $(\frac{1}{2}, 0)$. Students "see" this intersection through plots of $\theta = k + \phi$ for various k values and conjecture the intersection as having coordinates $(\frac{1}{2}, 0)$. However, they need to develop a proof.

When $k = \frac{\pi}{2}$ the rectangular plot of the biangular relation $\phi = \theta + \frac{\pi}{2}$ is the hyperbola:

$$\left(x - \frac{1}{2}\right)^2 - y^2 = \frac{1}{4}\,. \tag{3}$$

To prove this, we need show that the rectangular coordinates from (1),

$$P = \left(\frac{\cos(\theta)\sin(\phi)}{\sin(\theta + \phi)}, \frac{\sin(\theta)\sin(\phi)}{\sin(\theta + \phi)} \right),$$

with $\phi = \theta + \frac{\pi}{2}$ satisfy (3). We do this in *Mathematica*. Upon substituting (1) into (3) we obtain:

$$\left(\cos^2(\theta)\sec(2\theta) - \frac{1}{2}\right)^2 - \cos^2(\theta)\sec^2(2\theta)\sin^2(\theta) = \frac{1}{4}\,. \tag{4}$$

Applying *Mathematica*'s `TrigExpand` command to the left hand side of (4), or working by hand, we verify that (3) is satisfied.

Circle (pole-centered): $\phi = -2\theta$. Surprisingly, $\phi = -2\theta$ produces a circle, the unit circle centered on pole B (see Figure 8).

The more general biangular relations $\phi = -2\theta + k\pi$, where k is an integer, always produce the same circle, $(x-1)^2 + y^2 = 1$. In Figure 8, $\theta = \angle PAB$ and $\phi = \angle PBA$. Since $\overline{AB} = \overline{BP}$, $\angle APB = \theta$ also. Thus $\phi = \angle PBA = \pi - 2\theta$.

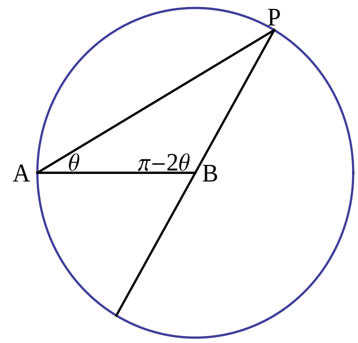


Figure 8: Figure for the biangular relation $\phi = -2\theta$.

Conversely if $\phi = -2\theta + k\pi$, where k is an integer, then, converting to rectangular coordinates, the expression $(x(\theta) - 1)^2 + y(\theta)^2$ reduces to 1, showing that points satisfying the biangular relations $\phi = -2\theta + k\pi$, where k is an integer, lie on the circle $(x(\theta) - 1)^2 + y(\theta)^2 = 1$.

Hyperbola (pole as focus): $\phi = 2\theta$. The relation $\phi = 2\theta$ describes a hyperbola, $9(x - \frac{1}{3})^2 - 3y^2 = 1$ (See Figure 9.) Pole $B = (1, 0)$ is a focus while the other focus is $(-\frac{1}{3}, 0)$.

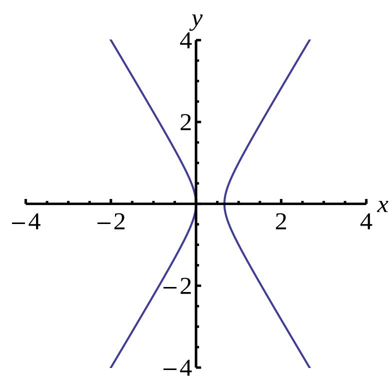


Figure 9: The biangular relation $\phi = 2\theta$.

Angel: $\phi = -2\theta + k$, $k \neq 0$. What about $\phi = -2\theta + k$, for $k \neq n\pi$, where $n = 0, 1, 2, \ldots$? In particular, we examine $k = \frac{\pi}{2}$. The curve $\phi = -2\theta + \frac{\pi}{2}$ is a right strophoid (see Figure 10). Originally named for their resemblance to a sword belt, these curves appear as loops with sweeping wings (see Figure 11). We will call them "angels", a more colorful name contrasting better with their duals, the devils described next.

Devil: $\phi = 2\theta + k$. The visual "opposite" of angels, these curves even resemble their namesake. Devils contain no loops; their "heads" have been turned inside out and look like horns (See Figure 12).

General Linear Function: $\phi = m\theta + k$, $m > 2$. The greater the coefficient of θ, the more

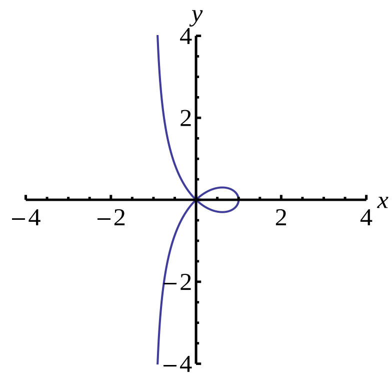


Figure 10: The right strophoid $\phi = -2\theta + \frac{\pi}{2}$.

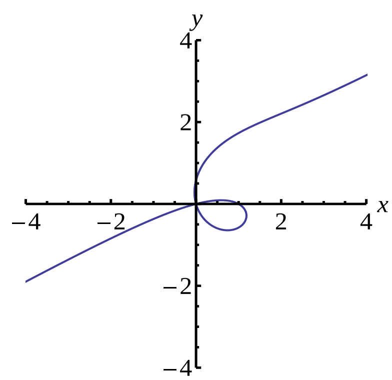


Figure 11: An angel $\phi = -2\theta + \frac{\pi}{6}$.

asymptotes in the figure and thus the more "arms" and "spikes" in the plot. It is interesting to see what happens to the plot of $\phi = -2\theta + k$ as k approaches 0. Figure 13 shows six curves of angels, $\phi = -2\theta + k$ with $k = 0, .01, .02, .03, .04, .05$, illustrating how an angel degenerates into a circle.

STRAIGHT LINES

A natural question to ask is "What biangular relation gives the biangular plot which is a straight line?" We give a characterization, essentially due to T. Biggin **[2]**. We refer to Figure 14 in our discussion.

We seek a biangular relation between θ and ϕ that will produce the line through P and Q at an angle α as measured counterclockwise off the polar axis $\overline{AB}$.

Using the Law of Sines in $\triangle APQ$, we have

$$\frac{\overline{AQ}}{\sin(\alpha - \theta)} = \frac{\overline{PQ}}{\sin(\theta)}\,. \tag{5}$$

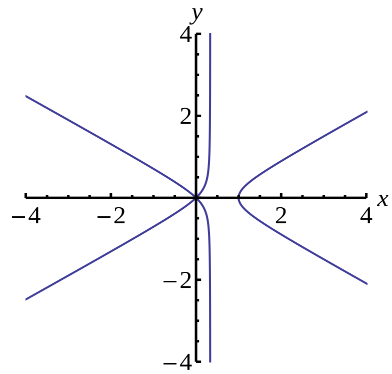


Figure 12: The devil $\phi = 2\theta + \frac{\pi}{2}$.

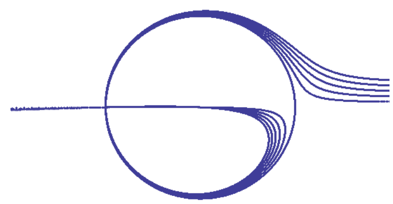


Figure 13: Six angels degenerating into a circle.

Similarly, in $\triangle BPQ$ we obtain

$$\frac{\overline{BQ}}{\sin(\pi - \alpha - \phi)} = \frac{\overline{PQ}}{\sin(\phi)}\,. \tag{6}$$

Using the trigonometric identity $\sin(\pi - x) = \sin(x)$, we can solve Equations (5) and (6) for $\overline{PQ}$ and set the results equal to each other. Upon inverting both sides of the resulting equation, we can use the identities $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$ to determine the following equation:

$$\frac{\sin(\alpha)\cos(\theta) - \cos(\alpha)\sin(\theta)}{\overline{AQ}\sin(\theta)} = \frac{\sin(\alpha)\cos(\phi) + \cos(\alpha)\sin(\phi)}{\overline{BQ}\sin(\phi)}\,.$$

Upon separation and applying the definition of cotangent, $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we obtain

$$\frac{\cot(\theta) - \cot(\alpha)}{\overline{AQ}} = \frac{\cot(\phi) + \cot(\alpha)}{\overline{BQ}}\,,$$

which can be written as

$$\frac{\cot(\theta) - c}{a} = \frac{\cot(\phi) + c}{b}\,,$$

where $c = \cot(\alpha)$, $a = \overline{AQ}$, and $b = \overline{BQ}$. This can then be written as $b(\cot(\theta) - c) = a(\cot(\phi) + c)$ or

$$b\cot(\theta) - a\cot(\phi) = ac + bc = c = \cot(\alpha)\,. \tag{7}$$

We see in Equation (7) that the familiar rectangular form of the line becomes a linear relationship between the cotangents of the respective angles θ and ϕ. In the early papers [**5,** p. 157] on biangular coordinates these cotangents *were* the biangular coordinates.

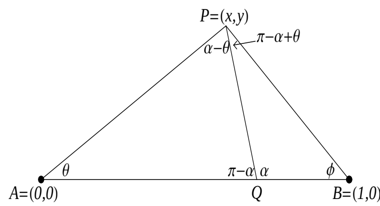


Figure 14: Derivation of biangular relation for a line through P and Q at angle α to the polar axis.

BIANGULAR RELATIONS WHICH PRODUCE THE PLOT OF A GIVEN RECTANGULAR RELATION

Consider the following question:

> For a given rectangular coordinate curve, $y = h(x)$, or relation, $F(x, y) = 0$, can we find a biangular relation, $G(\theta, \phi) = 0$, that will produce the given curve as its biangular plot?

Now for example, we can find the biangular relation which will generate the circle defined by the equation $x^2 + y^2 = r^2$. We find necessary and sufficient conditions on $f(\theta)$ so that the plot of the associated biangular relation, $\phi = f(\theta)$, is this circle. Indeed, the function $\phi = f(\theta)$ will generate the curve $x^2 + y^2 = r^2$ as a biangular relation if and only if

$$f(\theta) = f_{\text{top}} = \tan^{-1}\left(\frac{r\,\sin(\theta)}{1 - r\,\cos(\theta)}\right) \quad \text{or} \quad f(\theta) = f_{\text{bottom}} = \tan^{-1}\left(\frac{-r\,\sin(\theta)}{1 + r\,\cos(\theta)}\right).$$

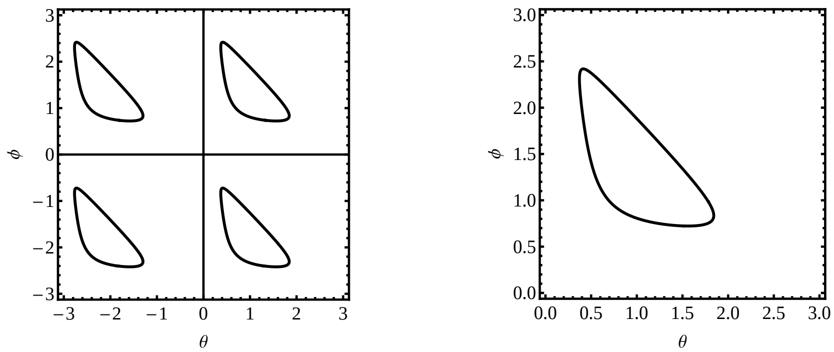
A similar method works for any curve with polar equation $r = r(\theta)$.

For a given rectangular relation, $F(x, y) = 0$, we may *not* be able to recover an explicit biangular relation, but we can use *Mathematica*'s `ContourPlot` command to show us the biangular relation. For example, suppose we wish to find the biangular relation that produces the plot of the circle $(x - 1)^2 + (y - 2)^2 = (1.5)^2$. We would use the $(x(\theta, \phi), y(\theta, \phi))$ coordinates as defined in (1) in the rectangular relation $(x - 1)^2 + (y - 2)^2 = (1.5)^2$ and implicitly plot

$$(x(\theta, \phi) - 1)^2 + (y(\theta, \phi) - 2)^2 = (1.5)^2\,, \tag{8}$$

in the (θ, ϕ) coordinate plane to see the biangular relation.

When we `ContourPlot` Equation (8) in *Mathematica* we see many (periodic) biangular relations (Figure 15a) each of which gives rise to the prescribed rectangular relation $(x-1)^2+(y-2)^2 = (1.5)^2$. In Figure 15b we zoom in on one of these whose analytic form we do not know, but whose shape we can see. This means that if we picked a point (θ, ϕ) on the plot of the biangular relation in Figure 15b its corresponding rectangular point would be on the given circle and each point on the given circle would come from some point on the biangular plot in Figure 15b. Of course, there would be an infinite number of preimages for the points on the given circle some of which are shown in Figure 15a.



(a) (b)

Figure 15: (a) An implicit plot of the many (periodic) biangular relation(s) which give rise to the rectangular relation $(x-1)^2 + (y-2)^2 = (1.5)^2$ and (b) a zoom in on one of these relations.

APPLICATION IN STUDY OF MAGNETS

Determining the lines of force of a bar magnet (see Figure 16) with equal charge at poles $A = (0, 0)$ and $B = (1, 0)$ is a natural object for study and can be expressed nicely in biangular coordinates. We present a notationally easier development in rectangular coordinates with a final relationship showing the simplicity in biangular coordinates.

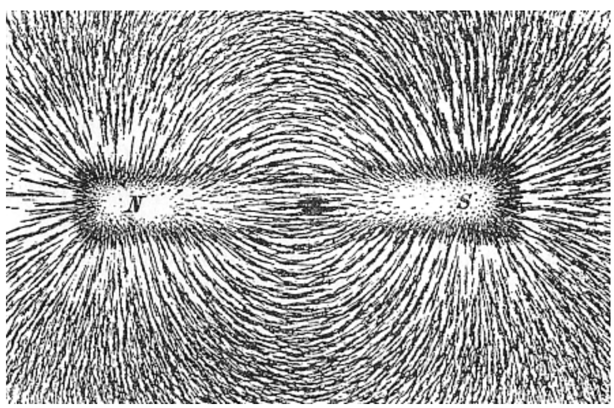


Figure 16: Magnetic lines of force of a bar magnet shown by iron filings on paper.[**11**]

To produce a compact biangular relation which describes the lines of force we compute (in rectangular coordinates) the sum of the forces acting on a unit charge at a point $Q(x, y)$ from equal charges placed at poles $A = (0, 0)$ and $B = (1, 0)$. There is a nice derivation of the relationship between x and y in [**4,** pp. 68-69] and it leads to a differential equation whose solution is implicit:

$$\frac{x}{\sqrt{x^2+y^2}} - \frac{1-x}{\sqrt{(1-x)^2+y^2}} = c\,,$$

for some constant c. Converting to biangular coordinates θ and ϕ we have:

$$\cos(\theta) - \cos(\phi) = c\,.$$

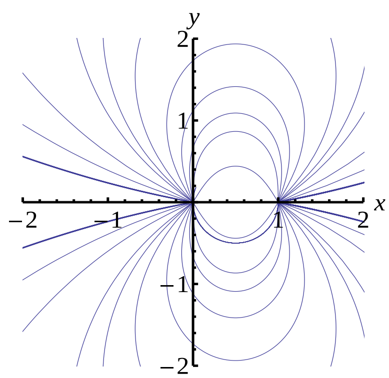


Figure 17: Biangular plots of $\cos(\theta) - \cos(\phi) = c$ for several values of c. Notice how these model the magnetic lines of force found in Figure 16.

We plot this mathematical model of the magnetic lines of force using this result, $\cos(\theta) - \cos(\phi) = c$, for a number of values of c in Figure 17 and see the similarity to Figure 16.

CONCLUSION

Biangular coordinates elegantly describe some curves whose representations are clumsy in other systems. The value of their study however lies not so much in what this system can accomplish, but more in what it can teach us about shapes, functions, and coordinate systems in general. The exploration of biangular coordinates captures the spirit and joy of mathematical discovery and study as it leads one into terrain that is both familiar and fresh.

References

[1] H. W. Baeumler, *Biangular Coordinates*, M.A. Thesis, University of Buffalo, 1950.

[2] T. Biggin, On biangular coordinates, and an extension of the system to space of three dimensions. *Quarterly Journal of Pure and Applied Mathematics*. **25** (1891) 237-258.

[3] L. N. M. Carnot, *Géométrie de Position* J. B. M. Dupart, Paris, 1803.

[4] A. Cohen, *An Elementary Treatise on Differential Equations*, D. C. Heath and Company, Boston, 1933.

[5] R. W. Genese, On a system of coordinates, *Proc. of the London Math. Soc.*, **S1-12** (1880) 157-168.

[6] ____________, On biangular coordinates, *Q. J. of Pure and App. Math.*, **18** (1882) 150-154.

[7] ____________, Biangular coordinates, In *Companion to the Weekly Problem Papers*, J. J. Milne, ed., Macmillan and Co., London, 1888, 78-98.

[8] D. Nelson, Bipolar coordinates and plotters, *PRIMUS*, **4** (1994) 77-83.

[9] I. Newton, *The Method of Fluxions and Infinite Series: With its Application to the Geometry of Curve-Lines* (trans. J. Colson), Henry Woodfall, London, 1736.

[10] W. Walton, On biangular coordinates, *Q. J. of Pure and App. Math.*, **9** (1868) 47-57.

[11] Wikipedia, (2007), `http://en.wikipedia.org/wiki/Image:Magnet0873.png`.

[12] G. B. M. Zerr, Biangular coordinates, *Amer. Math. Monthly*, **17** (1910) 34-38.

ACKNOWLEDGMENTS

The second author is grateful to Professor David Nelson for his introductory article **[8]** and to him and John Thoo for critiques of an early draft of this paper. He is also grateful for the many students in the past years who have elected to join in the fun of investigating biangular relations: Chris O'Malley, James Poylio, and Mark Lancaster at Rose-Hulman Institute of Technology and Brad Baggett, Richard Johnson, Booker Young, Ann Millen, and Ramiro Lopez at the United States Military Academy. All of these students have provided energy, inspiration, and ideas. We appreciate the careful attention to detail, comments, and suggestions of the referees and the editor.