The Hardest Straight-In Pool Shot

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This article started out as a napkin in a local establishment for imbibing and gaming in Shreveport, Louisiana. On the napkin was scribbled a diagram and some mathematics. The napkin then found itself thumb-tacked to the wall next to a pool table in said venue, the better for its authors to discuss any potential practical merits of the scribblings. Sadly, the napkin did not survive another two weeks, being removed either by an overzealous barkeeper, the tavern's owner, or one of the mathematically averse customers (of which there were many). Many years later, the napkin was missed and this article is its embellished reincarnation.

The mathematics here is elementary. Beyond basic trigonometry and elementary differential calculus, the only specialized knowledge required pertains to the geometric consequences of elastic collisons of spheres of uniform density. Some success in a laboratory component of the latter would be helpful. In other words, the usual first-year American barroom calculus sequence suffices as background. Familiarity is assumed with the basic jargon; terms such as “cue stick,” “billiard ball,” and “limit” will be used freely.

Let us set the scene more precisely. You are playing pool with a colleague or student, say a game of “8-ball,” and your opponent scratches. (To *scratch* is to commit a foul such that the cue ball is taken out of play, resulting in the situation of *ball-in-hand*. This typically occurs when the cue ball errantly lands in a pocket, but also when the cue ball (even more errantly) completely leaves the table itself. Only the two-dimensional aspects of pool will be considered here, so we need not consider the latter situation.) The rules dictate that after your opponent scratches, you freely place the cue ball by hand for your next shot. Well, not entirely freely, but you have some latitude—even quite a bit of longitude—you can place your ball anywhere you like between the *head cushion* and the *head string* (see Figure 1) and shoot at any object ball situated entirely between the head string and the *foot cushion*. It often happens that you then have an unobstructed shot at a ball and a clear path to the pocket. You decide, as do most players, to shoot straight into a pocket by placing the cue ball exactly in line with the pocket and the object ball.

Experienced players agree that such a straight-in shot is relatively easy if the object ball is either extremely close to the cue ball or extremely close to the pocket. This very idea, that both extremes should be easy, surprises some less experienced players, many of whom assume or feel that shot can only be

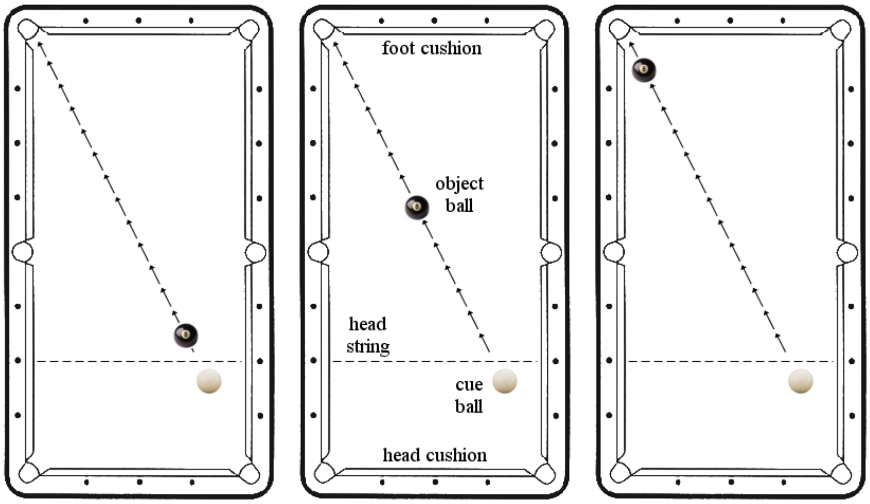

Figure 1: Straight-in shots to a corner pocket: easy, harder, easy. Where is the object ball for the hardest shot?

easier if the cue ball is nearer the object ball. Some of them have the opposite opinion, that the cue ball being too near the object ball presents more difficulty. Even more interesting is that experienced players tend to agree that of the two easy extremes, one is easier than the other. (Which one?) This conundrum leads to the first of two main questions we want to address here.

Question 1. Which of the two extreme situations (object ball very near the cue ball versus very near the pocket) is easier, or are they equally easy?

The second question was the subject of the lost napkin. If the two extremes pertaining to Question 1 have limiting difficulties of zero (we will have to make the notion of "difficulty" precise), then there is an intermediate distance of maximum difficulty. Is it half-way? Perhaps, but the geometry is not symmetric, so perhaps not. Two-thirds? Something involving the golden ratio? (Pleeease, let it be so.)

Question 2. What distance from the cue ball to the object ball makes the shot most difficult?

How difficult is a shot?

We need a "measure of difficulty." This is not easily agreed upon, but let's say you are sufficiently skilled that your errors are associated only with the physical aiming of the cue stick. We quantify this error as the angle between two lines—the line through (the centers of) the cue ball and the pocket, and the line of travel of the cue ball after it is struck. Letting θ be this (nonnegative) angle, we can compute the resulting deviation of the object ball from the pocket as it nears its target. The greater the deviation (for a given θ), the greater the difficulty.

To simplify the problem, forget about pockets and consider the situation shown in Figure 2. The cue ball and object ball have equal radii r. The cue ball is initially centered at $A = (0, -2r)$ and the object ball at $B = (0, y)$, where $0 \le y \le 1$ and $0 < r < 1/2$. (Notice that when $y = 0$, the cue ball and object ball are touching.) You aim to have the cue ball strike the object ball dead-center in order to send the center of the object ball across the target point $D = (0, 1)$, but your aim is slightly off by an angle θ. Letting C denote the center of the cue ball at the moment of its impact with the object ball, θ is equal to the measure of $\angle BAC$. After impact, the line of travel of the object ball is along the line through C and B. Of course, we assume that θ is small enough so that the cue ball hits the object ball, that is, $\sin\theta \leq \frac{2r}{y+2r}$. Let $x = x(y, \theta, r) = DE$ be the horizontal deviation when the object ball crosses the line $y = 1$ at E. We choose x as our measure of difficulty, the choice being natural, albeit somewhat arbitrary. (Other choices make sense, for example, the length of arc given by DB times $\angle DBE$.) It is clear that $x = 0$ if $y = 0$ or 1, independent of θ and r, so for each fixed θ and r, x will attain a maximum value for some y in $(0, 1)$.

Finding a formula for x is an easy exercise in trigonometry.

Fact 1. *The deflection x is given by*

$$x(y, \theta, r) = (1 - y) \tan \left(\sin^{-1} \left(\left(1 + \frac{y}{2r} \right) \sin \theta \right) - \theta \right),$$

for $0 < r < 1/2$, $0 \leq y \leq 1$, *and* $0 \leq \theta \leq \sin^{-1}(2r/(y + 2r))$.

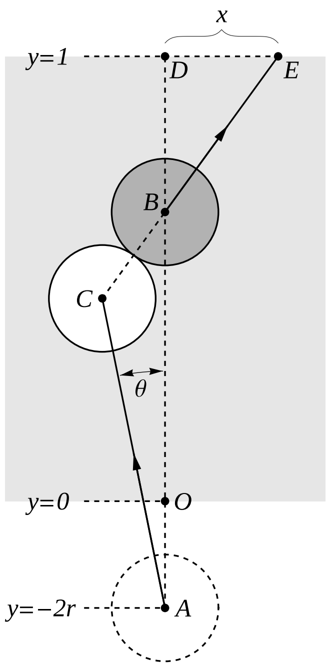

Figure 2: The moment of contact. Shooting for D but hitting at E.

[Editor's Note: Proofs of all Facts are posted on napkins at the CMJ/MAA web site, *MAA Online*, available at `http://www.maa.org/pubs/pool.html`.]

Answering Question 2

The next fact is an easy consequence of the previous one, needing only some familiarity with calculus. Here we consider r and θ to be fixed, with θ small.

Fact 2. *For $\theta \approx 0$,*

$$x \approx \left(\frac{(1-y)y}{2r}\right)\theta. \tag{1}$$

More precisely, $\lim_{\theta\to 0} \frac{(1-y)y}{2r}\frac{\theta}{x(y,\theta,r)} = 1.$

The next fact follows easily, and answers Question 2.

Fact 3. *For $\theta \approx 0$, the approximate deviation x given in* (1) *is maximized when $y \approx 1/2$.*

Again, a more precise statement involves a limit; the preceeding applies only asymptotically, as $\theta \to 0$. We will discuss more realistic values of θ in due course. An even more realistic approach is to consider θ as a small-valued random variable, and we briefly address that, too. A caveat about realism, though: our considerations are purely mathematical, taking into account none of the interesting physics at work in real pool playing (spin, banking, friction, etc.) For the real deal the reader should consult [**1**] and [**2**].

Meanwhile, it is interesting (but not surprising) that Fact 3 puts the object ball exactly midway between the extreme values of y in our scenario, as $\theta \to 0$.

No golden ratio ⟨sigh⟩.

Having so easily answered Question 2, we briefly carom off to consider the generalization to "combination shots." Suppose that n object balls are placed in sequence at positions $(0, y_k)$, with $0 < y_1 < y_2 < \cdots < y_n < 1$, where $y_{k+1} - y_k > 2r$ for $1 \leq k < n$ (so that the balls do not intersect). The goal is to shoot the cue ball into the first object ball, which then strikes the second, and so on, until the last (n^{th}) ball crosses the line $y = 1$. The hope is that the balls all move along the same straight line toward the point $(0, 1)$.

Let x_n denote the deviation of the n^{th} ball at $y = 1$. Using the same elementary approximations as employed earlier, it can be shown that for small errors in the initial angle θ, the greatest value of x_n occurs when the balls are evenly spaced.

Fact 4. *Fix r in $(0, \frac{1}{n+1})$. Then for $\theta \to 0$, the maximum deviation x_n corresponding to a straight-in n-ball combination shot occurs for a common separation $\Delta y = \frac{1-(n-1)2r}{n+1}$, giving $y_k = k\Delta y + (k-1)2r = \frac{k-(n-2k+1)(2r)}{n+1}$. For $\theta \approx 0$, this gives a deviation*

$$x_n \approx \left(\frac{1-(n-1)2r}{(n+1)2r} \right)^{n+1} 2r\theta. \tag{2}$$

So how hard is the two-ball combo compared with the single-ball case? The previous estimate gives $\frac{x_2}{x_1} \approx \frac{4}{27} \cdot \frac{(1-2r)^3}{2r}$. (Note that x_1 is equivalent to x in the original case of one ball, and that setting $n = 1$ correctly reduces Fact 4 and its development to that.) American barroom pool balls have radii of about $1.125''$, and pool tables measure about $46'' \times 92''$ (cramped taverns) or $50'' \times 100''$ (classier joints) [**5**]. The length of the longest possible shot shown in Figure 1 is thus about $83''$ for the typical, smaller barroom pool table. Scaling this to $y = 1$ gives a corresponding value of r we denote by $\tilde{r}$, where $\tilde{r} \approx .014$. With $r = \tilde{r}$, we get $\frac{x_2}{x_1} \approx 5.0$, meaning that on the shorter tables, the hardest two-ball combination shot is about five times as hard as the hardest single-ball shot of the same length.

It seems reasonable that the value of x_n increases with n, but in our scenario this is only true up to a point. This can be seen by considering the quantity being exponentiated in (2), which becomes less than 1 for large n, or by considering a limiting case of all n balls touching from cue ball to pocket, in which case the error x_n is zero. (For $r = \tilde{r}$, this nearly occurs when $n = 39$.) We therefore expect x_n to have a maximum for each r. For $r = \tilde{r}$ this occurs when $n = 7$ or 8, each of these values giving a relative difficulty $\frac{x_n}{x_1}$ of around 140 as $\theta \to 0$. The ratio $\frac{x_n}{x_1}$ (again, for $r = \tilde{r}$ and as $\theta \to 0$) is closest to 1 when $n = 15$ and decreases quickly for larger n. (See Table 1.) So (if you dare), try a 16-ball straight-in combination shot against your opponent's measly midway single-ball shot. (Um, it's best not to "bank" on that—check the math before wagering any large sums.)

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
D_n	5.0	17.	42.	79.	120	140	140	120	81.	47.	23.	9.1	3.1	.89	.21	.041

Table 1: Relative difficulty $D_n := \lim_{\theta \to 0} \frac{x_n}{x_1}$ for $r = \tilde{r}$.

We now return to the case of one object ball, as we consider the more realistic situation of "large" values of θ, in order to finally address Question 1.

It turns out that the situation is increasingly asymmetrical with increasing θ, the most difficult shot occurring when y is somewhat greater than the midway value of 1/2. This can be shown in the following way. Fix θ and r and let $f(y) = x(y, \theta, r)$. Using Fact 1, one can prove the following two facts.

Fact 5. *The function $x = f(y)$ is concave down.*

Fact 6. *The unique maximum of $f(y)$ occurs for $y > 1/2$.*

Fact 5 can be verified by showing that $f''(y) < 0$. Since $f(0) = f(1) = 0$, there is a single maximum value of x. Fact 6 can then be proven by showing that $f'(1/2) > 0$. (See Figure 3.)

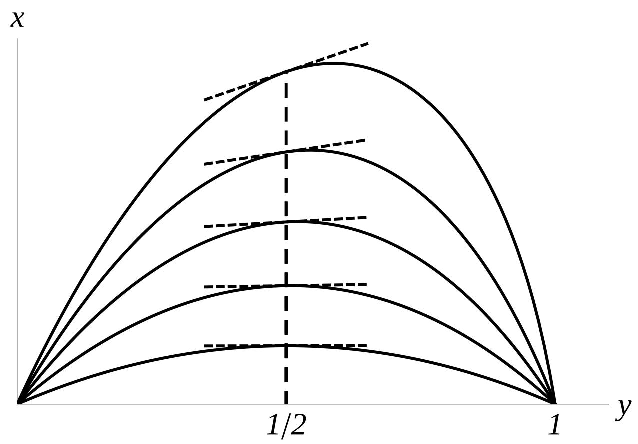

Figure 3: Plots of $x = f(y)$, with $r = .05$ and $\theta = 1°, 2°, 3°, 4°, 5°$.

Answering Question 1

We turn now to the relative difficulty of the two "easy" types of shot corresponding to $y \approx 0$ (the "near" shots with the object ball near the cue ball) and $y \approx 1$ (the "distant" ones with the object ball near the pocket). Figure 3 shows a curve that is steeper at $y = 1$ than at $y = 0$. If this is true for all admissible θ and r, then $f(u) < f(1-u)$ for sufficiently small u, showing that the difficulty is indeed asymmetric, with distant shots being harder than near ones, at least for object balls sufficiently near the cue ball or pocket (and, lest we forget, for our adopted measure of difficulty). This is confirmed by the next fact.

Fact 7. *If* $0 < r \leq 1/2$*, then* $f'(0) < -f'(1)$*.*

Verifying that $\sin^{-1}((a+1)\sin\theta) > \tan^{-1}(a\,\tan\theta) + \theta$ for $a = 1/2r$ and $0 < \sin\theta < 1/(a+1)$ establishes Fact 7. Furthermore, the following, stronger statement holds.

Fact 8. $f(1-u) > f(u)$ *for* $0 < u < 1/2$*.*

The verification of Fact 8 is significant, for it removes the caveat of "sufficiently small u" in judging the answer to Question 1. That, in turn, allows us to consider the incorporation of random variables. Certainly, even the best player makes small errors, but not always the same ones! The angle θ should vary over some range of values, each shot producing a different θ. If we assume $\tau(\theta)$ denotes a probabilty density function for θ (a smooth τ makes sense), then the expected value of x is given by $\bar{x}(y) = \int_0^{\theta_0} x(y,\theta,r)\,\tau(\theta)\ d\theta$ for each $y \in [0,1]$ (and fixed r; θ_0 is some arbitrary upper bound on θ). We may use Leibniz' rule to differentiate under the integral, obtaining $\bar{x}''(y) = \int_0^{\theta_0} \frac{\partial^2}{\partial y^2} x(y,\theta,r)\,\tau(\theta)\ d\theta$. (Many advanced calculus texts treat Leibniz' rule. See [**4**, p. 597] for a friendly approach; see [**3**] for a more complete story.) We have already noted that $f''(y) < 0$, so the integrand is everywhere negative, hence $\bar{x}(y)$ is concave down. It is clear that $\bar{x}(0) = \bar{x}(1) = 0$. And since $f'(y) > 0$ for $0 \le y \le 1/2$, we also have that $\bar{x}'(y) = \int_0^{\theta_0} \frac{\partial}{\partial y} x(y,\theta,r)\,\tau(\theta)\ d\theta > 0$ for $0 \le y \le 1/2$. This means that the expected value of x, just as with f, has a unique maximum at some value of y greater than $1/2$. The asymmetry holds—shots are harder somewhere past the midpoint.

Extreme pool

Our final investigation concerns extreme values of x.

Question 3. How large can x be?

Let $\theta_{\max}$ denote the largest value of θ that makes sense for all y in $(0,1)$, namely, $\theta_{\max} = \sin^{-1}(2r/(1+2r))$. A value of $\theta > \theta_{\max}$ corresponds to a shooter who will not be able to make the longest shot ($y = 1$), and we hereby retire such players.

Clearly, for unrestricted θ, there is no theoretical maximum for x so long as the player at least strikes the object ball and r is allowed to be arbitrarily small. Extremely small values of r may be unrealistic, but even for the reasonable value of $r = .02$, which corresponds to shots at about half the length of a barroom pool table, the maximum x is around six lengths of the shot. Anyway, we have already eliminated any such poor shooters from consideration by requiring $\theta \leq \theta_{\max}$.

For fixed y and r, x is obviously an increasing function of θ, so we consider the maximum displacement $m(y,r) := x(y, \theta_{\max}, r)$. For a given r, what is the maximum of $m(y,r)$ for $0 \leq y \leq 1$? A numerical calculation gives the maximum of $m(y,\tilde{r}) \approx .29$, occurring for $y \approx 0.61$. (Recall that $\tilde{r}$ is our scaled "realistic" value of r corresponding to a typical barroom table.) Just for fun, consider very small r. (This is equivalent to fixing r and letting the size of the table grow large.) Since x is also an increasing function of $1/r$ (for fixed y and θ), there is a maximum M of x for $0 \leq \theta \leq \theta_{\max}$, $0 < r < 1/2$ and $0 < y < 1$, namely

$$M = \lim_{r \to 0^+} \max_{0<y<1} m(y,r).$$

Fact 9. *For fixed r, the function $m(y,r)$ has a unique maximum on the interval $0 \leq y \leq 1$. If y_r is where it occurs, then*

$$\lim_{r \to 0^+} y_r = 1/\phi,$$

where ϕ is the golden ratio!

Well I'll be snookered.

Honestly, the appearance of ϕ is entirely serendipitous. There was no intentional mining, panning, or digging for gold. In any event, it shows that the most difficult shot for the most error-prone viable shooter occurs when y is close to $1/\phi = .61803\cdots$, and that the least upper bound of the deviation is $M = 0.30028\cdots$.

Conclusion

After all this, we readily concede that our model needs more scrutiny. For one thing, there is more to making a good pool shot than aligning the stick—one must to judge the point of contact accurately. But in this case, too, it can be argued that θ is still the critical quantity. In that sense, perhaps our approach applies somewhat realistically, regardless of whether the player can't shoot straight or can't see straight. But not to be ignored is that straight-in shots are the easiest of the shots presented to pool shooters. Our treatment, especially of combination shots, is highly idealized, since all the shooter needs to do is aim for the center of the first ball, period. In practice, however, a combination shot is virtually certain not to be straight-in, and so the shooter must then judge the many points of contact between balls, produced in succession, as well as actually hitting the correct spot. Another issue is that a successful shot does not require pinpoint accuracy, as the pockets are larger than the balls (typically by a factor of about two). This actually compensates *more* for errors on longer shots than on smaller ones. Are the longer shots still more difficult in this case? Incorporating such complex realities might make for interesting projects. Analyzing the effect of slight errors in the placement of balls for a straight-in combination shot would also be enlightening and might lead to results that are testable in the pool hall.

On that note, perhaps someone reading this will have an itch to substantiate (or refute) our claims by trial and error. Line 'em up then. Chalk up the cue and let the scratching begin!

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