GRAPHS AND ZERO-DIVISORS

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In an algebra class, one uses the zero-factor property to solve polynomial equations. For example, consider the equation $x^2 = x$. Rewriting it as $x(x-1) = 0$, we conclude that the solutions are $x = 0, 1$. However, the same equation in a different number system need not yield the same solutions. For example, in $\mathbb{Z}_6$ (the integers modulo 6), not only 0 and 1, but also 3 and 4 are solutions. (Check this!)

In a commutative ring R (like $\mathbb{Z}_6$), an element r is a *zero-divisor* if there exists a nonzero $s \in R$ such that $rs = 0$. In $\mathbb{Z}_6$ the zero-divisors are 0, 2, 3, and 4 because $0 \cdot 2 = 2 \cdot 3 = 3 \cdot 4 = 0$. A commutative ring with no nonzero zero-divisors is called an *integral domain*. The zero-factor property used in high school algebra holds for integral domains, but does not hold for *all* commutative rings. Because of this, ring theorists find zero-divisors very interesting.

In general, the set of zero-divisors lacks algebraic structure. In particular, the set of all zero-divisors of a ring R, denoted $Z(R)$, is not always closed under addition. In $\mathbb{Z}_6$, we see that 2 and 3 are zero-divisors, but $2+3$ is not. Hence, $Z(R)$ is typically not a subring and thus also not an ideal. Recently, a new approach to studying the set of zero-divisors has emerged from an unlikely direction: graph theory.

In this paper we present a series of projects that develop the connection between commutative ring theory and graph theory. These are suitable for a student who has completed an introductory undergraduate abstract algebra course. Projects not marked with an asterisk are straight-forward and should require well less than

one week of work. Projects marked with one or two asterisks are more challenging undertakings requiring weeks or even months of dedicated effort. As a general reference on ring theory, we recommend [**8**]; for graph theory we recommend [**7**].

Zero-Divisor Graphs

A 1988 a paper by Istvan Beck [**6**] set out to establish "a connection between graph theory and commutative ring theory which will hopefully turn out to be mutually beneficial for these two branches of mathematics." Beck's original definition of the graph of a commutative ring was modified by D.F. Anderson and P. Livingston [**2**], and it is their definition that is now standard. Denote by $\Gamma(R)$ the graph whose vertices are the elements of $Z(R)^*$, the *nonzero* zero-divisors of a commutative ring R, and in which two vertices a and b are connected by an edge, denoted $a - b$, if and only if $ab = 0$. We call $\Gamma(R)$ the *zero-divisor graph of* R.

Knowledge of properties of the zero-divisor graph sometimes provides a surprising amount of algebraic insight into a ring. This is investigated in the projects below. We begin with some definitions. A graph Γ is *connected* if one can move along a *path* of edges from any one vertex to any other vertex. The *distance* between two distinct vertices a and b, denoted $d(a, b)$, is the minimum number of edges required to move from one vertex to the other, if such a path exists. If there is no path between the two vertices, then the distance between them is defined to be infinite. The *diameter* of a graph is the maximal length of all minimal paths between vertices. A *cycle* is a path of the form $a_0 - a_1 - \cdots - a_n - a_0$ where $a_i \neq a_j$ for $i \neq j$. The *girth* of a graph Γ is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise, the girth is defined to be infinite. Finally, we say

there is *a loop at* x if there is an edge $x - x$. If Γ is the zero-divisor graph of a commutative ring R, then a loop at x is equivalent to saying $x^2 = 0$.

Project 1. *Construct* $\Gamma(R)$ *for several commutative rings* R*. Determine the diameter and girth of each graph you construct.*

A student should begin by constructing zero-divisor graphs for $\mathbb{Z}_n$ for several different values of n, then expand to direct products of such rings and factor rings of polynomial rings. As an aid to building these graphs, several Mathematica notebooks which constuct zero-divisor graphs for rings of the form $\mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}_m$, and $\mathbb{Z}_n[x]/(f(x))$ are downloadable from **[10]**. The interested undergraduate can also create notebooks to construct the zero-divisor graph of other rings.

The reason that zero-divisor graphs have so captured the attention of ring theorists is the remarkable amount of graph-theoretic structure they exhibit. The next project reveals some of this structure.

Project 2. **Based on the graphs constructed in Exercise 1, conjecture whether* $\Gamma(R)$ *is always connected or not. Make a conjecture about the diameter and the girth of* $\Gamma(R)$*. Prove your conjectures.*

$\Gamma(R)$ and Ideals

One fundamental question is, "When does $Z(R)$ form an ideal?" The next project gives an algebraic answer.

Project 3. *Prove that* $Z(R)$ *is an ideal if and only if* $Z(R)$ *is closed under addition.*

The following project asks the student to discover what graph-theoretic properties of $\Gamma(R)$ will guarantee that $Z(R)$ forms an ideal.

Project 4. ***What graph properties of $\Gamma(R)$ imply that $Z(R)$ is an ideal? Consider separately finite rings of diameter $0, 1, 2$ and 3. Does any of your work generalize to infinite rings?*

It may be helpful to know that $Z(R)$ is finite if and only if either R is finite or an integral domain. It is not known if there is a graph-theoretic method, of determining whether $Z(R)$ forms an ideal when $\Gamma(R)$ is an infinite, diameter 3 graph. This does not absolve the researcher from finding such a method, but it does represent a challenge.

Though $Z(R)$ is not always an ideal, $Z(R)$ can always be expressed as a union of prime ideals [**9**, p. 2]. When R is a finite commutative ring, $Z(R)$ is, in fact, the union of at most *two* prime ideals. This leads to the next project.

Project 5. ***Suppose $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are prime ideals. How may the elements in these ideals be determined from the graph $\Gamma(R)$?*

To illustrate the nature of this question, if $x \in Z(R)^*$, the *annihilator* of x is the set of y such that $xy = 0$. In graph theory terms, it is the set of vertices adjacent to x in $\Gamma(R)$ and, if there is a loop at x (meaning that $x^2 = 0$), x itself. In other words, the annihilator is the neighborhood of the vertex x in $\Gamma(R)$. Project 5 asks if there such a description of the elements of each prime ideal.

Realizable Zero-Divisor Graphs

When can a given graph can be realized as the zero-divisor graph of some commutative ring? There are some easy results in this direction. Recall that a graph G is *complete bipartite* if it can be partitioned into two disjoint, nonempty vertex

sets P and Q such that two vertices a and b are connected by an edge if and only if $a \in P$ and $b \in Q$. Any complete bipartite graph $B\,(p-1, q-1)$ where p and q are prime is the zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$. On the other hand, no graph Γ of diameter 7 is the zero-divisor graph of a commutative ring.

A *star graph* is a complete bipartite graph in which one of the two disjoint vertex sets contains exactly one element, called the *center* of the graph. For example, the zero-divisor graph of $\mathbb{Z}_2 \times \mathbb{F}_4$ (where $\mathbb{F}_4$ is the field of four elements) is a star graph with center $(1, 0)$. (Check this!)

As an undergraduate, Warfel [**14**] discovered, however, that not every star graph can be realized as a zero-divisor graph, namely if G is a star graph with a non-looped center, but a non-central vertex in G is looped, then G cannot be $\Gamma(R)$ for any commutative ring R. The requirement that the center not be looped is seen to be necessary by examing $\Gamma(\mathbb{Z}_9)$. Other results on this topic can be found in [**1**], [**4**], and [**13**].

Project 6. **What other graph properties automatically disqualify a given graph Γ from being realizable as a zero-divisor graph of a commutative ring?*

This is a very open-ended project with many answers. Certainly, there are diameter and girth limitations; other characteristics you may want to consider are looped end vertices, cliques, and cut-vertices.

The study of zero-divisors of a commutative ring via its zero-divisor graph is still in its infancy and continues to be a rapidly expanding area of interest in commutative ring theory. In the ten years since [**2**] appeared, over 100 research articles have appeared in professional journals and undergraduate publications. Though

somewhat costly computationally, the construction of $\Gamma(R)$ often reveals a great deal about the algebraic structure of $Z(R)$. The following amazing result illustrates this. Two finite, reduced commutative rings which are not fields are ring-isomorphic if and only if their zero-divisor graphs are graph-isomorphic [**1**, p. 69]. This stunning theorem demonstrates that occasionally $\Gamma(R)$ contains *all* essential information about R.

The authors would like to thank the referee and, in particular, the editor for greatly improving this article.

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