Some Graphical Solutions of the Kepler Problem

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1. INTRODUCTION. When a particle $P$ moves in an inverse square acceleration field, its orbit is a conic section. If the acceleration is repulsive—that is, of the form $\mu \| \vec{F}\vec{P} \|^{-3} \vec{F}\vec{P}$ for a positive constant $\mu$ and a fixed point $F$—then the orbit is a branch of a hyperbola with $F$ at the focus on the convex side of the branch. If the acceleration is attractive ($-\mu \| \vec{F}\vec{P} \|^{-3} \vec{F}\vec{P}$), then the orbit is a circle with $F$ at the center, an ellipse or parabola with $F$ at a focus, or a branch of a hyperbola with $F$ at the focus on the concave side of the branch. We refer to such orbits as Keplerian orbits after Johannes Kepler, who proposed them as models of planetary motion.

It takes rather special conditions to set up circular or parabolic orbits, so it is probably safe to say that in Nature most closed orbits are ellipses, and most nonclosed orbits are hyperbolas. Ironically, these are precisely the cases in which it is impossible to formulate the position of $P$ as an elementary function of time $t$—the very thing we most want to know!

Nevertheless, for any given time $t$ it is possible to locate $P$ using graphs of elementary functions, thereby achieving a partial victory. This endeavor has an interesting history in the case of elliptical orbits, where locating $P$ is called the “Kepler problem.” It involves a famous equation known as “Kepler’s equation,” which we discuss later. In his book Solving Kepler’s Equation over Three Centuries Peter Colwell says [5, p. ix]:

In virtually every decade from 1650 to the present there have been papers devoted to the Kepler problem and its solution. We can see from a list of them that the problem has enticed a wide variety of scientists to comment on or involve themselves in its solution.

[T]he Kepler problem has acquired an undeniable luster and allure for the modern practitioner. Any new technique for the treatment of transcendental equations should be applied to this illustrious test case; any new insight, however slight, lets its concealer join an eminent list of contributors.

Colwell’s exhaustive survey of the subject includes power series solutions, iterative procedures, and graphical solutions. Names from the history of graphical solutions include those of Kepler himself, John-Dominique Cassini (namesake of the “Cassini division” in the rings of Saturn) and his son Jacques, Johann Franz Encke (namesake of Comet Encke and the “Encke gap” in Saturn’s rings), Christopher Wren, John Wallis, and Isaac Newton [5].

This impressive history suggests that new graphical solutions should have something extra to recommend them. In this article we present an approach that gives double solutions for both elliptic and hyperbolic orbits, includes both attractive and repulsive accelerations, and leads to interesting animations linking Keplerian motion with other physical processes. Throughout the article we encourage the reader to look at and enjoy these animations, which appear on the website [9].

2. THE COSINE SOLUTIONS. We begin with two particularly simple solutions and derive forms of Kepler’s equation in their proofs.
Proposition 1. Let $E$ be the ellipse with equation

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \quad (0 < b < a),$$

eccentricity $\epsilon$, and foci $F_1 = (0, a\epsilon)$ and $F_2 = (0, -a\epsilon)$. Fix a positive parameter $\mu$, and let $\gamma_t$ signify the cosine curve moving to the right at constant speed $v = (b/\epsilon)^{1/2} \sqrt{\mu/a^3}$ whose equation at time $t$ is

$$y = a \cos((\epsilon/b)(x - vt)). \quad (1)$$

Then, at any time $t$, $E \cap \gamma_t = \{P_1(t), P_2(t)\}$, where the points $P_1$ and $P_2$ have the following properties:

(i) $P_1$ orbits anticlockwise relative to $F_1$, with acceleration $-\mu ||F_1 P_1||^{-3} F_1 P_1$;

(ii) $P_2$ orbits clockwise relative to $F_2$, with acceleration $-\mu ||F_2 P_2||^{-3} F_2 P_2$.

![Figure 1. Illustration of Proposition 1.](image)

Proposition 2. Let $H$ be the upper branch of the hyperbola with equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

eccentricity $\epsilon$, and foci $F_1 = (0, a\epsilon)$ and $F_2 = (0, -a\epsilon)$. Fix a positive parameter $\mu$, and let $\Gamma_t$ signify the hyperbolic cosine curve moving to the right at constant speed $v = (b/\epsilon)^{1/2} \sqrt{\mu/a^3}$ whose equation at time $t$ is

$$y = a \cosh((\epsilon/b)(x - vt)). \quad (2)$$

Then, at any time $t$, $H \cap \Gamma_t = \{P_1(t), P_2(t)\}$, where the points $P_1$ and $P_2$ have the following properties:

(i) $P_1$ orbits anticlockwise relative to $F_1$, with acceleration $-\mu ||F_1 P_1||^{-3} F_1 P_1$;

(ii) $P_2$ orbits clockwise relative to $F_2$, with (repulsive) acceleration $\mu ||F_2 P_2||^{-3} F_2 P_2$.

The proofs of Propositions 1 and 2 depend on facts from basic orbital mechanics, including the following. Consider a fixed point $F$ and an attractive acceleration field $-\mu ||F X||^{-3} F X$ for $X \neq F$. Any ellipse with $F$ as a focus—and any branch of a hyperbola with $F$ as a focus on the concave side—is a possible orbit for a point $P$ subject
to the given acceleration. Similarly, given a repulsive acceleration field $\mu \| \dot{F} \dot{X} \|^{-3} \dot{F} \dot{X}$, any branch of a hyperbola with $F$ as a focus on the convex side is a possible orbit for a point $P$. In either case, the motion of $P$ is completely determined if we specify the field, the orbit curve, the direction of motion along the curve, and the initial position $P(0)$. For example, the field and the orbit curve determine the energy of the orbit, which in turn determines the speed of $P$ as a function of its position.

Proof of Proposition 1. By the previous remarks there exist position functions $P_1(t)$ and $P_2(t)$ satisfying (i) and (ii), respectively. To specify the two we want, put $P_1(0) = P_2(0) = (0, a)$. It remains to show that $E \cap \gamma_i = \{ P_1, P_2 \}$ for all times $t$. By our choice of $P_1$ and $P_2$, they can be parameterized as

$$P_1 = (-b \sin E_1, a \cos E_1), \quad P_2 = (b \sin E_2, a \cos E_2), \quad (3)$$

where $E_1$ and $E_2$ are strictly increasing functions of time $t$ with $E_1(0) = E_2(0) = 0$. For $i = 1, 2$ define $\mathbf{r}_i = \dot{F}_i \dot{P}_i$ and $\mathbf{r}_i = d\mathbf{r}_i/dt$. The positions and velocities are then

$$\mathbf{r}_1 = (-b \sin E_1, a \cos E_1 - a \varepsilon), \quad \mathbf{r}_2 = (b \sin E_2, a \cos E_2 + a \varepsilon),$$

$$\dot{\mathbf{r}}_1 = (-b \dot{E}_1 \cos E_1, -a \dot{E}_1 \sin E_1), \quad \dot{\mathbf{r}}_2 = (b \dot{E}_2 \cos E_2, -a \dot{E}_2 \sin E_2).$$

It is well known that

$$\| \mathbf{r}_i \times \dot{\mathbf{r}}_i \| = \sqrt{\mu a (1 - \varepsilon^2)}$$

(for example, see [6, p. 122]). Since $b = a \sqrt{1 - \varepsilon^2}$, we have

$$\| \mathbf{r}_i \times \dot{\mathbf{r}}_i \| / (ab) = \sqrt{\mu / a^3}.$$

Computing cross products, we obtain

$$\sqrt{\mu / a^3} = \left| \frac{\| \mathbf{r}_1 \times \dot{\mathbf{r}}_1 \|}{ab} \right| = \dot{E}_1 - \varepsilon \dot{E}_1 \cos E_1 = \frac{d}{dt} (E_1 - \varepsilon \sin E_1)$$

and

$$\sqrt{\mu / a^3} = \left| \frac{\| \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \|}{ab} \right| = \dot{E}_2 + \varepsilon \dot{E}_2 \cos E_2 = \frac{d}{dt} (E_2 + \varepsilon \sin E_2).$$

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Using the initial values $E_1(0) = E_2(0) = 0$ we find that

$$\sqrt{\mu/a^3} t = E_1 - \varepsilon \sin E_1, \quad \sqrt{\mu/a^3} t = E_2 + \varepsilon \sin E_2.$$  \hspace{1cm} (4)

The quantity $M := \sqrt{\mu/a^3} t$ is known as the mean anomaly. We can write the equations in (4) in terms of $M$ as

$$M = E_1 - \varepsilon \sin E_1$$ \hspace{1cm} (5)

and

$$M = E_2 + \varepsilon \sin E_2.$$ \hspace{1cm} (6)

Equation (5) is Kepler’s equation for elliptical orbits, and $E_1$ is known as the eccentric anomaly. We can write the equations in (4) in terms of $M$ as

$$M = E_1 - \varepsilon \sin E_1$$ \hspace{1cm} (5)

and

$$M = E_2 + \varepsilon \sin E_2.$$ \hspace{1cm} (6)

We now fix $t$ in $\mathbb{R}$ and show that $\mathcal{E} \cap \gamma_t = \{P_1, P_2\}$. Since $vt = (b/\varepsilon)\sqrt{\mu/a^3} t = (b/\varepsilon)M$, we can rewrite (1) as

$$y = a \cos ((\varepsilon/b)x - M).$$ \hspace{1cm} (7)

Now define $\phi = (\varepsilon/b)x - M$, so that

$$x = (b/\varepsilon)(M + \phi).$$ \hspace{1cm} (8)

Then by (7) and (8) a point $(x, y)$ on $\gamma_t$ has the form $((b/\varepsilon)(M + \phi), a \cos \phi)$. Consequently, each point of $\mathcal{E} \cap \gamma_t$ has the form $(b \sin \alpha, a \cos \alpha)$ for some real number $\alpha$ satisfying

$$(b \sin \alpha, a \cos \alpha) = ((b/\varepsilon)(M + \phi), a \cos \phi).$$ \hspace{1cm} (9)

Equating $y$-components in (9) we have $\cos \alpha = \cos \phi$, so either $\sin \alpha = \sin \phi$ or $\sin \alpha = -\sin \phi$.

If $\sin \alpha = \sin \phi$, the intersection point has the form $Q_1 = (b \sin \phi, a \cos \phi)$, and comparing $Q_1$ with the right side of (9) gives $(b/\varepsilon)(M + \phi) = b \sin \phi$. Hence

$$Q_1 = (b \sin \phi, a \cos \phi) \Rightarrow M = -\phi + \varepsilon \sin \phi.$$  \hspace{1cm} (10)

The latter equation is just (5) with $E_1$ replaced by $-\phi$, so the unique solution is $\phi = -E_1$. In this instance, $Q_1 = (-b \sin E_1, a \cos E_1) = P_1$.

If $\sin \alpha = -\sin \phi$, the intersection point has the form $Q_2 = (-b \sin \phi, a \cos \phi)$, and comparing $Q_2$ with the right side of (9) gives $(b/\varepsilon)(M + \phi) = -b \sin \phi$. Accordingly,

$$Q_2 = (-b \sin \phi, a \cos \phi) \Rightarrow M = -\phi - \varepsilon \sin \phi.$$  \hspace{1cm} (11)

The latter equation is (6) with $E_2$ replaced by $-\phi$, yielding the unique solution $\phi = -E_2$. As a result, $Q_2 = (b \sin E_2, a \cos E_2) = P_2$. \hspace{1cm} \blacksquare

**Remark.** Although we could have argued the case for $P_2$ by symmetry, we structured the proof as we did in order to abbreviate the proof in the hyperbolic case.
Proof of Proposition 2 We sketch the proof, which exactly parallels that of Proposition 1. By a similar argument there exist position functions \( P_1(t) \) and \( P_2(t) \) satisfying (i') and (ii'), respectively, with \( P_1(0) = P_2(0) = (0, a) \). It remains to show that \( \mathcal{H} \cap \Gamma_t = \{ P_1, P_2 \} \).

The curve \( \mathcal{H} \) can be parameterized as \( \mathcal{H} = \{(b \sinh \alpha, a \cosh \alpha) : \alpha \in \mathbb{R}\} \), so in this case the vectors \( r_1 := \frac{\overrightarrow{F_1P_1}}{e} \) and \( r_2 := \frac{\overrightarrow{F_2P_2}}{e} \) can be parameterized as

\[
  r_1 = (b \sinh H_1, a \cosh H_1 - a \varepsilon), \quad r_2 = (b \sinh H_2, a \cosh H_2 + a \varepsilon),
\]

where \( H_1 \) and \( H_2 \) are strictly increasing functions of time \( t \) and \( H_1(0) = H_2(0) = 0 \). In the hyperbolic case it is well known that

\[
  \| r_1 \times \dot{r}_1 \| = \| r_2 \times \dot{r}_2 \| = \sqrt{\mu(e^2 - 1)}
\]

(for example, see [6, p. 122]). Since \( b = a\sqrt{e^2 - 1} \) for a hyperbola, we again have

\[
  \| r_1 \times \dot{r}_1 \|/(ab) = \sqrt{\mu/a^3}.
\]

Thus we can compute \( \sqrt{\mu/a^3} \) in terms of \( H_1 \), and again in terms of \( H_2 \), to arrive at equations analogous to (5) and (6):

\[
  M = \varepsilon \sinh H_1 - H_1, \quad M = \varepsilon \sinh H_2 + H_2,
\]

where once more \( M = \sqrt{\mu/a^3} t \). These are Kepler's equations for hyperbolic orbits, with the first better known than the second. In a manner analogous to the proof of Proposition 1, the proof can be finished by showing that \( \mathcal{H} \cap \Gamma_t = \{ P_1, P_2 \} \) at each time \( t \).

Remark. For the rest of the article we concentrate on elliptical orbits. We abbreviate the label of the point \( P_1 \) in Proposition 1 to \( P \), and define \( F = F_1 \) and \( E = E_1 \). Formally, for given \( \varepsilon \) in \( (0, 1) \) and \( M \) in \( [0, \pi] \), the Kepler problem for elliptical orbits is to solve Kepler's equation \( M = E - \varepsilon \sin E \) explicitly for \( E \) in \( [0, \pi] \). We can do this graphically by using just the left half of \( \mathcal{E} \) and assuming that the unit of length has been chosen so that \( a = \varepsilon/\sqrt{1 - \varepsilon^2} \) and \( b = \varepsilon \). Given \( M \) in \( [0, \pi] \), we intersect the half-ellipse with the appropriate half-cycle of \( y = a \cos(x - M) \) (see (7)) as in Figure 3. The distance from \( P \) to the dashed line is then \( \varepsilon \sin E + M = E \).

![Figure 3. Cosine solution of the Kepler problem.](image)

3. GENERATING SOLUTIONS. An immediate consequence of Proposition 1 is that if the ellipse \( \mathcal{E} \) translates to the left with velocity \( v = (-b/\varepsilon)\sqrt{\mu/a^3}, 0) \) as \( P \)
orbits around $\mathcal{E}$, then the point $P + vt$ follows the cosine curve $\gamma_0$. We can exploit this idea to generate more graphical solutions (not necessarily double ones) of the Kepler problem. Following convention, we define the mean motion $n$ by $n = \sqrt{\mu/a^3}$, so that $M = nt$. Let $v$ be the constant velocity vector

$$v = (kn, 0) \quad (k \neq 0).$$

Then $vt = (k, 0)M = (k, 0)(E - \varepsilon \sin E)$. Since $P = (-b \sin E, a \cos E)$, we then have

$$P + vt = (\tilde{b} \sin E + kE, a \cos E),$$

where

$$\tilde{b} = -b - k\varepsilon.$$

Imitating Proposition 1, we can now use the curve

$$\gamma(v) := \{P + vt : t \in \mathbb{R}\} = \{(\tilde{b} \sin E + kE, a \cos E) : E \in \mathbb{R}\},$$

by dragging it with velocity $-v$ across the stationary ellipse $\mathcal{E}$. As we do so, the intersection point $P$ will execute Keplerian motion around $\mathcal{E}$ with the upper focus $F$ as the center of acceleration. In particular, the curve $\gamma(v)$ can be used in a graphical solution of the Kepler problem analogous to Figure 3.

For “nice” solutions we prefer the curve $\gamma(v)$ to be well known, like the cosine curve in Proposition 1 (the case $k = -b/\varepsilon$). Two more such curves are the prolate cycloid and the curtate cycloid. Normally we think of these curves as being generated by rolling wheels (see [15] and [16] for a review) but planets can make them, too.

The prolate cycloid. If $k = (a - b)/\varepsilon$, then $\tilde{b} = -a$ and by (12) the curve

$$\gamma(v) = \{(-a \sin E + [(a - b)/\varepsilon]E, a \cos E) : E \in \mathbb{R}\}$$

is a prolate cycloid (i.e., a cycloid with loops). This follows from the fact that $(a - b)/\varepsilon < a$. We leave the proof as an exercise, with the reminder that $b = a\sqrt{1 - \varepsilon^2}$. An illustration of this trajectory appears in Figure 5, and an animated version appears on the website [9].

The curtate cycloid. If $k = -(a + b)/\varepsilon$, then $\tilde{b} = a$ and the curve in (12) is

$$\gamma(v) = \{(a \sin E - [(a + b)/\varepsilon]E, a \cos E) : E \in \mathbb{R}\},$$

which is a curtate cycloid (i.e., a cycloid with no loops or cusps). This derives from the fact that $(a + b)/\varepsilon > a$.

The curtate cycloid is often mentioned in hydrodynamics as a model of surface waves on deep water. In this model the surface fluid particles move in constant-speed circular orbits (see, for example, [10, pp. 421–422] or [7]). This makes for a nice solution of the Kepler problem in which a surface fluid particle represents the mean anomaly. An animation of this solution appears on the website [9].

In Proposition XXXI of the Principia of 1686, Isaac Newton gave a solution of the Kepler problem that used a curtate cycloid to locate the planet on the orbit ellipse [11, pp. 92–93]. His solution was based on an earlier one conceived by Christopher Wren.
and published by John Wallis in 1659 [5, p. 20]. However, Newton’s cycloid was not the same as ours, and he did not locate the planet by intersecting the cycloid with the orbit (see also section 6).

**Remark.** There is another interpretation of \( \gamma(v) \) worth mentioning. Observe that if we make the formal substitution \( E = n\tau \) in (11), we obtain

\[
P + vt = (\tilde{b}\sin(n\tau) + kn\tau, a\cos(n\tau)) = \tilde{P}(\tau) + vt,
\]

(13)

where

\[
\tilde{P}(\tau) = (\tilde{b}\sin(n\tau), a\cos(n\tau)).
\]

If we think of the new variable \( \tau \) as time, then \( \tilde{P}(\tau) \) is the position function of a linear-force pendulum in which the pendulum bob orbits around the ellipse \( \tilde{E} = \{\tilde{P}(\tau) : \tau \in [0, 2\pi)\} \), which is degenerate when \( \tilde{b} = 0 \). Using (10), (12), and (13) we can write

\[
\gamma(v) = \{\tilde{P}(\tau) + vt : \tau \in \mathbb{R}\}.
\]

Thus, when the “planetary” ellipse \( E \) and the pendulum ellipse \( \tilde{E} \) translate together moving parallel to their horizontal axes, the planet and the pendulum bob follow the same curve \( \gamma(v) \) but at different speeds. It is tempting to think of these curves as “elliptical cycloids.”

4. **Binary Stars.** In the more general case when \( v \) may have nonzero \( y \)- and \( z \)-components, curves of the form \( \gamma(v) = \{P + vt : t \in \mathbb{R}\} \) arise in the study of binary stars. To a good approximation, the center of mass \( F \) of a binary system moves with a constant velocity \( v \) with respect to our sun. In the frame of reference of \( F \), each binary companion moves in an elliptical orbit with \( F \) at one focus, and the acceleration of a companion \( P \) has the form \(-\mu_p \|F \tilde{P}\|^{-3}F \tilde{P} \), where \( \mu_p \) is a positive constant depending on the masses of the two stars (for more details see [6, chap. 6]). If the orbit ellipses do not rotate—due to distortion of the shapes of the stars, for example—then their motion is pure translation at velocity \( v \). In this case the trajectory of each companion is a curve of the form \( \gamma(v) \).

In a famous example, the mathematician and astronomer Friedrich Wilhelm Bessel correctly deduced in 1844 that the bright star Sirius had a dimmer, unseen companion. The two stars are now known as Sirius A and B, respectively. Bessel based his conclusion on repeated observations of Sirius A, which showed that, as seen from our sun, its path was not a straight line but a wavy trajectory with a period of about fifty years (Figure 4). This type of double star—in which the trajectory of a visible star signals the presence of an invisible companion—is called an *astrometric binary*. The trajectories of Sirius A and B still make appearances in astronomical research [4].

The examples in the preceding section show that possible trajectories for binary stars include sine waves, prolate cycloids, and curtate cycloids. A particularly nice example occurs when the binary companions have equal masses and congruent orbit ellipses with \( b = a/2 \). In this case \( v \) can be chosen so that one star follows a prolate cycloid while the other follows a sine wave (Figure 5). A proof of this fact and an accompanying animation appear on the website [9]. For our purposes, this setup provides a different kind of double solution to the Kepler problem.
5. ANIMATING SOLUTIONS. With the first predicted return of Halley’s comet in 1758, scientists and science enthusiasts became interested in what were essentially animated solutions of the Kepler problem. At that time the motions of the known planets could be fairly well represented by constant-speed circular orbits. However, the orbit of Halley’s comet was unmistakably elliptical, and Kepler’s second law of planetary motion (the vector from the sun to the comet sweeps out equal areas in equal times) implied that the comet would move much more swiftly at perihelion than at aphelion. This revived interest in a device called a cometarium (see [2], [3], or [12, pp. 44–48]).

A cometarium (also called a “mercurium” or “equal area machine”) was typically a box with an elliptical track on one face and a crank on the side (Figure 6). As the crank was turned at a constant rate, an interior mechanism would push a miniature comet around the track, hopefully at correctly varying speeds with an impressive “whoosh” past the perihelion point. John Theophilus Desaguliers demonstrated the first such device to the assembled Fellows of the Royal Society in 1732. Unfortunately, his device and many of its descendants provided incorrect approximations to the true motion [3].

Our results show that neither complicated devices nor high-speed number crunching are necessary to achieve correct animations. For example, we can move the graph of $y = a \cos(\epsilon x / b)$ past the orbit ellipse $E$ at any constant speed to get realistic planetary
motion of both intersection points. A faster or slower speed merely indicates a larger or smaller value of the constant $\mu$, corresponding to a more or less massive sun at either focus. In the spirit of the cometarium, the reader is once again encouraged to enjoy animations of this and other results on the website [9].

6. CONCLUSION. As mentioned earlier, Newton’s curtate cycloid solution to the Kepler problem was different from the one presented here. In fact, Newton’s cycloid did not have the proper wavelength $2\pi (a + b)/\varepsilon$ to be generated by the planet with the orbit ellipse in uniform motion [11, pp. 92–93]. Similarly, R. A. Proctor [13] presented a complicated solution (equivalent to an earlier one by J. C. Adams [1]) involving the orbit ellipse and a cosine curve, but again the use was different and the curve did not have the proper wavelength $2\pi b/\varepsilon$ to be a sinusoidal trajectory of the planet. Thus others have come close to—but apparently missed—the simple fact that planets, binary stars, and other astronomical bodies can generate well-known curves when their orbits are in uniform motion.

These curves provide simple and apparently new graphical solutions to the Kepler problem. With regard to pedagogy, they make pleasing visual links between planetary motion and other physical phenomena, such as simple harmonic motion. In a related example, Donald Saari [14] observed that the trajectory of Mars when seen from Earth is well approximated by a limaçon with a loop, and he remarked that his students found this to be “a more persuasive illustration of the relevance of limaçons and cardioids than many of the standard examples used in calculus” [14, p. 106]. Certainly it would not hurt students to know—or even to derive—the fact that at least one body in the two-body problem (e.g., a binary star) will follow a sine wave, a prolate cycloid, or a curtate cycloid, provided the center of mass moves with the appropriate constant velocity. For the author, curves that can be generated by rolling wheels, translating pendulums, water waves, and binary stars are sufficient to inspire reflection on the interconnectedness of Nature, and of mathematics.
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