On the Converse of Lagrange's Theorem

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Undoubtedly the most basic result in finite group theory is the Theorem of Lagrange that says the order of a subgroup divides the order of the group. Herstein [**8**, p. 75] likens this theorem to the ABC's for finite groups. G. A. Miller [**9**, p. 23] calls it "the most important theorem of group theory" (see also [**2**, p. 130]).

Although it has been known since 1799 that the group A_4 consisting of the 12 even permutations on $\{1,2,3,4\}$ has no subgroup of order 6, it is surprising that a number of abstract algebra textbooks fail to mention that this most natural converse of the most important theorem of finite group theory is false (e.g. [**11**], [**1**]). Many authors mention the fact without proof (e.g. [**8**, p. 72]) or use phrases such as "A_4 can be shown to have no subgroup of order 6" (e.g. [**4**, p. 102], [**7**, p. 40]), perhaps giving students the impression that such a proof is omitted because it is too difficult. Some books (e.g. [**3**, p. 245]) give complicated proofs that A_4 has no subgroup of order 6. Most books that do provide a proof, do so long after introducing Lagrange's Theorem and invoke relatively sophisticated notions such as normality (e.g. [**2**, p. 142]), factor groups ([**5**, p. 151], [**12**, p. 104]), the classification of groups of order 6 ([**10**, p. 200]), conjugacy arguments ([**6**, p. 45]) or, in some cases, even Sylow's Theorem ([**3**, p. 245]).

It seems to have been overlooked that there is a simple argument requiring nothing more complicated than the basic properties of cosets to prove that A_4 has no subgroup of order 6. Before giving our argument we observe that $A_4 = \{(1),(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)\}$ contains eight elements of order 3.

Now suppose H is a subgroup of A_4 of order 6 and let a be any element of order 3. Then, since H has index 2, at most two of the cosets H, aH and a^2H are distinct. But the equality of any pair of these implies that $a \in H$. Thus, H contains all eight elements of order 3.

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