The Square Roots of 2 × 2 Matrices

DONALD SULLIVAN
University of New Brunswick
Fredericton, N.B., Canada

Introduction In a recent article MacKinnon [1] describes four methods that may be used to find square roots of 2×2 matrices. The first of these methods requires that the matrix for which the square roots are sought be diagonalizable and, subsequently, this method was used by Scott [2] to determine all the square roots of 2×2 matrices. A surprising conclusion is that scalar 2×2 matrices possess double-infinities of square roots whereas nonscalar 2×2 matrices have only a finite number of square roots.

The purpose of this article is to show how the Cayley-Hamilton theorem may be used to determine explicit formulae for all the square roots of 2×2 matrices. These formulae indicate exactly when a 2×2 matrix has square roots, and the number of such roots.

By definition, the square roots of a 2×2 matrix, A, are those 2×2 matrices, X, for which

$$X^2 = A. \tag{1}$$

However, for each square matrix X, the Cayley-Hamilton theorem states that

$$X^2 - (\operatorname{tr} X)X + (\det X)I = 0. \tag{2}$$

Thus, if a 2×2 matrix A has a square root X, then we may use (2) to eliminate X^2 from (1) to obtain

$$(\operatorname{tr} X)X = A + (\det X)I.$$

Further, since $(\det X)^2 = \det X^2 = \det A$, then $\det X = \varepsilon_1\sqrt{\det A}$, that is $\det\sqrt{A} = \varepsilon_1\sqrt{\det A}$, so that the above result simplifies to the identity:

$$(\operatorname{tr} X)X = A + \varepsilon_1\sqrt{\det A}\, I, \quad \varepsilon_1 = \pm 1. \tag{3}$$

Case 1: A is a scalar matrix. If A is a scalar matrix, $A = aI$, then (3) gives

$$(\operatorname{tr} X)X = (1+\varepsilon_1)aI, \quad \varepsilon_1 = \pm 1.$$

Hence, either $(\operatorname{tr} X)X = 0$ or $(\operatorname{tr} X)X = 2aI$. The first of these possibilities determines the general solution of (1) as

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad \alpha^2 + \beta\gamma = a \tag{4a}$$

and it covers the second possibility if $a = 0$. On the other hand, if $a \neq 0$ then the second possibility, $(\operatorname{tr} X)X = 2aI$, implies X is scalar and has only the pair of solutions

$$X = \pm\sqrt{a}\, I. \tag{4b}$$

For this case we conclude that if A is a zero matrix then it has a double infinity of square roots as given by (4a) with $a = 0$, whereas if A is a nonzero, scalar matrix then it has a double-infinity of square roots plus two scalar square roots as given by (4a) and (4b).

Case 2: A is not a scalar matrix. If A is not a scalar matrix then $\operatorname{tr} X \neq 0$ in (3). Consequently, every square root X has the form:

$$X = \tau^{-1}\left(A + \varepsilon_1\sqrt{\det A}\, I\right), \quad \tau \neq 0$$

Substituting this expression for X into (1) and using the Cayley-Hamilton theorem for A we find

$$\begin{aligned} A^2 + \left(2\varepsilon_1\sqrt{\det A} - \tau^2\right)A + (\det A)I &= 0 \\ ((\operatorname{tr} A)A - (\det A)I) + \left(2\varepsilon_1\sqrt{\det A} - \tau^2\right)A + (\det A)I &= 0 \\ \left(\operatorname{tr} A + 2\varepsilon_1\sqrt{\det A} - \tau^2\right)A &= 0. \end{aligned}$$

Since A is not a scalar matrix then A is not a zero matrix, so

$$\tau^2 = \operatorname{tr} A + 2\varepsilon_1\sqrt{\det A}, \quad (\tau \neq 0, \varepsilon_1 = \pm 1). \tag{5}$$

If $(\operatorname{tr} A)^2 \neq 4 \det A$ then both values of ε_1 may be used in (5) without reducing τ to zero. Consequently, it follows from (3) that we may write X, the square root of A, as

$$X = \varepsilon_2 \frac{A + \varepsilon_1\sqrt{\det A}\, I}{\sqrt{\operatorname{tr} A + 2\varepsilon_1\sqrt{\det A}}}. \tag{6a}$$

Here each $\varepsilon_i = \pm 1$, and if $\det A \neq 0$ the result determines exactly four square roots for A. However, if $\det A = 0$ then result (6a) determines two square roots for A as given by

$$X = \pm \frac{1}{\sqrt{\operatorname{tr} A}} A. \tag{6b}$$

Alternatively, if $(\operatorname{tr} A)^2 = 4 \det A \neq 0$, then one value of ε_1 in (5) reduces τ to zero whereas the other value yields the results, $2\varepsilon_1\sqrt{\det A} = \operatorname{tr} A$ and $\tau^2 = 2 \operatorname{tr} A$. In this case, A has exactly two square roots given by

$$X = \pm \frac{1}{\sqrt{2 \operatorname{tr} A}}\left(A + \frac{1}{2}(\operatorname{tr} A)I\right). \tag{6c}$$

Finally, if $(\operatorname{tr} A)^2 = 4 \det A = 0$ then both values of ε_1 reduce τ to zero in (5). Hence it follows by contradiction that A has no square roots.

For this case we conclude that a nonscalar matrix, A, has square roots if, and only if, at least one of the numbers, $\operatorname{tr} A$ and $\det A$, is nonzero. Then the matrix has four square roots given by (6a) if

$$(\operatorname{tr} A)^2 \neq 4 \det A, \ \det A \neq 0$$

and two square roots given by (6b) or (6c) if

$$(\operatorname{tr} A)^2 \neq 4 \det A, \ \det A = 0 \quad \text{or} \quad (\operatorname{tr} A)^2 = 4 \det A, \ \det A \neq 0.$$

It is worth noting from (6a) that

$$\operatorname{tr} X = \operatorname{tr} \sqrt{A} = \varepsilon_2\sqrt{\operatorname{tr} A + 2\varepsilon_1\sqrt{\det A}}.$$

Hence using the identity, $\det\sqrt{A} = \varepsilon_1\sqrt{\det A}$ as applied in (3), result (6a) may be rewritten as

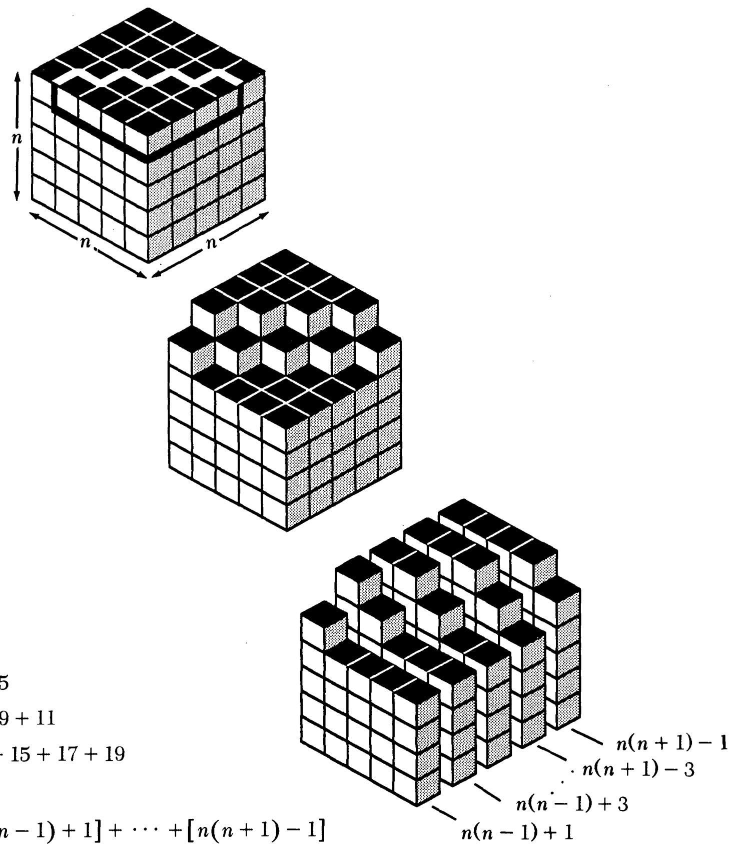
$$\sqrt{A} = \frac{1}{\operatorname{tr}\sqrt{A}}\left(A + \det\sqrt{A}\, I\right),$$

which is equivalent to the Cayley-Hamilton theorem for the matrix $\sqrt{A}$. This same deduction can be made, of course, for all other cases under which $\sqrt{A}$ exists.

REFERENCES

1. Nick MacKinnon, Four routes to matrix roots, *Math. Gaz.* 73 (1989), 135–136.
2. Nigel H. Scott, On square-rooting matrices, *Math. Gaz.* 74 (1990), 111–114.
3. Howard W. Eves, *Elementary Matrix Theory*, Dover Publications, Mineola, NY, 1980.
4. Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis*, Camb. University Press, NY, 1990.

Proof without Words: Every Cube Is the Sum of Consecutive Odd Numbers



$1^3 = 1$

$2^3 = 3 + 5$

$3^3 = 7 + 9 + 11$

$4^3 = 13 + 15 + 17 + 19$

$\vdots$

$n^3 = [n(n-1)+1] + \cdots + [n(n+1)-1]$

—Roger B. Nelsen
Lewis and Clark College
Portland, OR 97219