Reaching for Common Ground in K-12 Mathematics Education

Deborah Loewenberg Ball, Joan Ferrini-Mundy, Jeremy Kilpatrick, R. James Milgram, Wilfried Schmid, Richard Schaar

Over the past decade, much debate has arisen between mathematicians and mathematics educators. These debates have significantly distracted the attention of key players at all levels, and have impeded efforts to improve mathematics learning in this country. This document represents an attempt to identify a preliminary list of positions on which many may be able to agree.

Our effort arose out of discussions between Richard Schaar and major players in both communities. He suspected that some of these disagreements might be more matters of language and lack of communication than representative of fundamental differences of view. To test this idea, he convened a small group of mathematicians and mathematics educators.[1]

We tried to bring clarity to key perspectives on K-12 mathematics education. We began by exploring typical "flashpoint" topics and probed our own positions on each of these to determine whether and where we agreed or disagreed. For the first meeting, held in December 2004, we began with summary statements drawn from prior exchanges among the members of our group. We affirmed some agreements in this meeting, and "discovered" others. We listened closely to one another, frequently asking for clarification, or for examples. We tested our understanding of others' points of view by proposing statements that we then examined collectively. We drafted this document as a group, composing actual text as we worked. One of us typed, and our emerging draft was projected onto a screen in the meeting room. The process enabled us to take issue with particular words and terms, and then reshape them until all of us were satisfied. We were forced to look closely at our own language and to seek common ground, not only in the terms we used, but even in their nuanced meaning.

This document was completed at our second meeting, in June 2005. All of us are encouraged by the extent of our agreements. The document treats only a subset of the controversial issues, many of which arise in K-8 mathematics. We expect to continue the process by examining a wider range of major issues in mathematics education. We have necessarily limited ourselves to questions depending primarily on disciplinary judgment, as opposed to those requiring empirical evidence.

We begin with three fundamental assertions and continue with a list of areas in which we found common ground. For each, we have written a short paragraph that captures the fundamental points of our agreement. Our next step is to explore how others respond to the document, and to use their responses to decide how best to make progress on the aims of this project. Our goal is to

[1]We are grateful to the National Science Foundation and Texas Instruments Inc. for funding this portion of our work.

forge new alliances, across communities, necessary to develop effective solutions to the serious problems that plague mathematics education in this country.

Fundamental Premises

All students must have a solid grounding in mathematics to function effectively in today's world. The need to improve the learning of traditionally underserved groups of students is widely recognized; efforts to do so must continue. Students in the top quartile are underserved in different ways; attention to improving the quality of their learning opportunities is equally important. Expectations for all groups of students must be raised. By the time they leave high school, a majority of students should have studied calculus.

1. Basic skills with numbers continue to be vitally important for a variety of everyday uses. They also provide crucial foundation for the higher-level mathematics essential for success in the workplace which must now also be part of a basic education. Although there may have been a time when being to able to perform extensive paper-and-pencil computations mechanically was sufficient to function in the workplace, this is no longer true. Consequently, today's students need proficiency with computational procedures. *Proficiency*, as we use the term, includes both computational fluency and understanding of the underlying mathematical ideas and principles.[2]

2. Mathematics requires careful reasoning about precisely defined objects and concepts. Mathematics is communicated by means of a powerful language whose vocabulary must be learned. The ability to reason about and justify mathematical statements is fundamental, as is the ability to use terms and notation with appropriate degrees of precision. By *precision*, we mean the use of terms and symbols, consistent with mathematical definitions, in ways appropriate for students at particular grade levels. We do *not* mean formality for formality's sake.

3. Students must be able to formulate and solve problems. Mathematical problem solving includes being able to (a) develop a clear understanding of the problem that is being posed; (b) translate the problem from everyday language into a precise mathematical question; (c) choose and use appropriate methods to answer the question; (d) interpret and evaluate the solution in terms of the original problem, and (e) understand that not all questions admit mathematical solutions and recognize problems that cannot be solved mathematically.

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[2]Kilpatrick, J., Swafford, J. and Findell, B. (Eds.). *Adding It Up: Helping Children Learn Mathematics*, Washington, DC: National Academy Press, 2001.

Areas of Agreement

Discussions of the following items are often riddled with difficulties in communication, making it sometimes confusing to determine whether and how much disagreement exists. Issues also arise from a confounding of a mathematical idea with its implementation in the classroom. For example, the fact that algorithms have often been taught badly does not imply that algorithms themselves are bad. We worked to clarify issues and terms and arrived at statements with which we agreed.

A. Automatic recall of basic facts: Certain procedures and algorithms in mathematics are so basic and have such wide application that they should be practiced to the point of automaticity. Computational fluency in whole number arithmetic is vital. Crucial ingredients of computational fluency are efficiency and accuracy. Ultimately, fluency requires automatic recall of basic number facts: by *basic number facts*, we mean addition and multiplication combinations of integers $0 - 10$. This goal can be accomplished using a variety of instructional methods.

B. Calculators: Calculators can have a useful role even in the lower grades, but they must be used carefully, so as not to impede the acquisition of fluency with basic facts and computational procedures. Inappropriate use of calculators may also interfere with students' understanding of the meaning of fractions and their ability to compute with fractions. Along the same lines, graphing calculators can enhance students' understanding of functions, but students must develop a sound idea of what graphs are and how to use them independently of the use of a graphing calculator.

C. Learning algorithms: Students should be able to use the basic algorithms of whole number arithmetic fluently, and they should understand how and why the algorithms work. Fluent use and understanding ought to be developed concurrently. These basic algorithms were a major intellectual accomplishment. Because they embody the structure of the base-ten number system, studying them can reinforce students' understanding of the place value system.

   More generally, an *algorithm* is a systematic procedure involving mathematical operations that uses a finite number of steps to produce a definite answer. An algorithm can be implemented in different ways; different recording methods for the same algorithm do not constitute different algorithms. The idea of an algorithm is fundamental in mathematics. Studying algorithms beyond those of whole number arithmetic provides opportunities for students to appreciate the diversity and importance of algorithms. Examples include constructing the bisector of an angle; solving two linear equations in two unknowns; calculating the square root of a number by a succession of dividing and averaging.

D. Fractions: Understanding the number meaning of fractions is critical. Ratios, proportions, and percentages cannot be properly understood without fractions. The arithmetic of fractions is important as a foundation for algebra.

E. Teaching mathematics in "real world" contexts: It can be helpful to motivate and introduce mathematical ideas through applied problems. However, this approach should not be elevated to a general principle. If all school mathematics is taught using real world problems, then some important topics may not receive adequate attention. Teachers must choose contexts with care. They need to manage the use of real-world problems or mathematical applications in ways that focus students' attention on the mathematical ideas that the problems are intended to develop.

F. Instructional methods: Some have suggested the exclusive use of small groups or discovery learning at the expense of direct instruction in teaching mathematics. Students can learn effectively via a mixture of direct instruction, structured investigation, and open exploration. Decisions about what is better taught through direct instruction and what might be better taught by structuring explorations for students should be made on the basis of the particular mathematics, the goals for learning, and the students' present skills and knowledge. For example, mathematical conventions and definitions should not be taught by pure discovery. Correct mathematical understanding and conclusions are the responsibility of the teacher. Making good decisions about the appropriate pedagogy to use depends on teachers having solid knowledge of the subject.

G. Teacher knowledge: Teaching mathematics effectively depends on a solid understanding of the material. Teachers must be able to do the mathematics they are teaching, but that is not sufficient knowledge for teaching. Effective teaching requires an understanding of the underlying meaning and justifications for the ideas and procedures to be taught, and the ability to make connections among topics. Fluency, accuracy, and precision in the use of mathematical terms and symbolic notation are also crucial. Teaching demands knowing appropriate representations for a particular mathematical idea, deploying these with precision, and bridging between teachers' and students' understanding. It requires judgment about how to reduce mathematical complexity and manage precision in ways that make the mathematics accessible to students while preserving its integrity.

   Well-designed instructional materials, such as textbooks, teachers' manuals, and software, may provide significant mathematical support, but cannot substitute for highly qualified, knowledgeable teachers. Teachers' mathematical knowledge must be developed through solid initial teacher preparation and ongoing, systematic professional learning opportunities.