Splines and Their Impact

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Duck

Draftsman’s Spline
The Mathematical Model

Minimize Energy \approx \int_a^b \kappa(t)^2 \approx \int_a^b [f''(t)]^2 \, dt

The Natural Cubic Spline

1) \( s \) is a piecewise cubic polynomial
2) \( s \in C^2[a, b] \)
3) \( s''(a) = s''(b) = 0 \)
http://www.cs.wisc.edu/~deboor/bib/bib.html
Journals

Constructive Approximation

Journal of Approximation Theory
• Computer-Compatible Functions
  • Store
  • Evaluate using + - * / <

• Polynomials

\[ p(x) = \sum_{i=0}^{d} c_i x^i \]

• Spline = Piecewise Polynomial
Polynomial Splines

• Finite dimensional space

• Stable local basis (B-splines)
• Easy to store, evaluate, and manipulate
  
  • Derivatives
  • Integrals
  • Inner-products

• Good approximation power

$$\|D^i(f - s)\|_p \leq C h^{m+1-i} \|D^{m+1}f\|_p$$

• Spline Functions: Basic Theory, Larry L. Schumaker, Wylie, 1981
Applications

• Interpolation
• Smoothing noisy data
• Approximation of functions
• Design of (parametric) curves
• Signal processing (wavelets)
• Image processing (edge detection)
• Solution of ODE’s
Parametric Spline with Control Polygon

Font Design
Generalized Splines

• Piecewise solution of ODE

• More general smoothness conditions between pieces

• Similar properties to polynomial splines

• Examples: trigonometric, hyperbolic, exponential splines
Cam Design
The Zoo of Splines

• Analytic splines
• Arc splines
• Beta splines
• B-splines
• Bernoulli splines
• Box splines
• Cardinal splines
• Circular splines
• Complete splines
• Complex splines
• Confined splines
• Deficient splines
• D^m splines
• Discrete splines
• Euler splines
• Exponential splines
• Gamma splines
• GB-splines
• HB-splines
• Hyperbolic Splines
• Monosplines
• Nu-splines
• Natural splines
• L-splines
• Lg-splines
• Nonlinear splines
• One-sided splines
• Parabolic splines
• Perfect splines
• Periodic splines
• Poly-splines
• Rational splines
• Simplex splines
• Spherical splines
• Taut splines
• Tchebycheffian splines
• Tension splines
• Thin plate splines
• Trigonometric splines
• Whittaker splines
Tensor Product Splines

\[ s(x) = \sum_{i=0}^{d} \sum_{j=0}^{d} c_{ij} B_i(x) B_j(y) \]
Bivariate Splines on Triangulations

\[ S^r_d(\Delta) := \{ s \in C^r(\Omega) : s|_T \in \mathbb{P}_d, \text{ for all } T \in \Delta \}, \]
Constructive Theory

• Dimension

• Stable local basis

• Approximation power

• Applications
Local Support

Approximation Power

\[ \| D_x^\alpha D_y^\beta (f - s) \|_{p,\Omega} \leq C |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,p,\Omega} \]
Applications

• Interpolation

• Smoothing noisy data

• Approximation of functions

• Design of (parametric) surfaces

• Image processing (compression, denoising)

• Solution of PDE’s

Trivariate Splines

• Piecewise polynomials on tetrahedral partition
• Constructive theory
• Volume modelling (CAT scans)
• Finite element method for PDE’s
• Approximation Theory
• Numerical Analysis
• Computer Science
• Application Areas

  Engineering
  Biosciences, Chemistry, Physics, Geophysics, Meteorology
  Medicine
  Business and Social Sciences
  Imaging and Visualization
  Computer-aided design and Manufacture
  Computer Vision and Robotics


