Preface

A Gentle Introduction to the American Invitational Mathematics Exam is a celebration of mathematical problem solving. It is written at the level of the high school American Invitational Mathematics Exam (AIME). The book is intended, in part, as a resource for comprehensive study and practice for the AIME competition to be used by students, coaches, teachers, and mentors preparing for the exam.

However, it is also written for those who enjoy solving problems, exploring recreational mathematics, and sharpening critical thinking and problem solving skills. It is not intended just for the exceptionally brilliant or for those who are intensely competitive, even though the AIME is a competition. Anyone who enjoys thinking about interesting problems in mathematics should enjoy this book.

One of my aims in writing this book was to make the problems accessible and interesting to a wide audience and to lure in those who feel perhaps that the AIME problems are not for them. Therefore, even if you are not involved in the AIME as a student or mentor, as long as you have an interest in mathematical problem solving and a desire to explore fun mathematical problems at the high school level, you should find this book engaging. You may surprise yourself with what you can do!

The AIME is the second in a series of exams known collectively as the American Mathematics Competitions (AMC). We will describe the series of exams comprising the AMC, including the focus of this book, the AIME, and offer some of advice and encouragement. After that, we will survey a large compendium of topics and problems from past AIMEs that will form the basis for learning, discussion, and discovery.

The American Mathematics Competitions

In 1950, the Mathematical Association of America launched the AMC as a means of "strengthening the mathematical capabilities of our nation's youth". Each year, over 350,000 students in roughly 6,000 schools participate in the MAA's AMC competitions.

The AMC consists of a series of examinations covering a broad spectrum of topics. The problems appearing in them can all be solved by using precalculus methods. The exams become progressively more difficult, beginning with the AMC 10 and AMC 12 in February

each year, followed by the AIME in March or early April, the United States of America Mathematical Olympiad (USAMO) later in April, and the International Mathematics Olympiad (IMO) in July[1]. With the help of a three-week Mathematical Olympiad Summer Program (MOSP) hosted at the University of Nebraska each June for the six members of the United States IMO team, the United States has performed exceptionally well in the IMO. The United States fielded its first IMO team in 1974, and placed second. The number of countries participating in the IMO is usually between 80 and 90. The United States has placed first in total team score at the IMO five times and, on 28 occasions, has finished in the top three.

The goals of the AMC, however, reach far beyond success in the IMO. There have been disturbing trends in the United States in recent years in mathematics education at the middle and high school levels. Many students in the U.S. underperform in mathematics compared to their counterparts in other countries. The AMC examinations are designed to involve students of all mathematical abilities, not just the brightest few. The AMC 10, AMC 12, and AIME contain problems that do not require extraordinarily creative thought. The hope is that they can serve as a confidence boost to many students, and that participation in the exam will foster enthusiasm for mathematics in general. The problems, by and large, are fun and engaging. They usually do not require heavy machinery or excessive computations. Instead, they call on a student to analyze ideas, to think logically, and in the case of some of the harder problems, to find a clever approach.

About the AMC 10, AMC 12, and the American Invitational Mathematics Exam

The AMC 10 exam is open to all students in 10th grade or below, while the AMC 12 is open to all students in 12th grade or below. Both exams consist of 25 multiple-choice problems to be solved without a calculator in 75 minutes. Students receive 6 points for each correct answer, 1.5 points for each blank answer, and no points for a wrong answer. The AMC 10 and AMC 12 exams are both offered twice every year. Therefore, an 11th or 12th grader can attempt the AMC 12 exam twice, while a 10th grader could attempt the AMC 10 twice, the AMC 12 twice, or take each exam once.

Students who score at least 120 points or finish in the top 2.5% on the AMC 10 exam or who score at least 100 points or finish in the top 5% on the AMC 12 exam are invited to take the American Invitational Mathematics Exam. The AIME is a 15-problem exam in which each answer is an integer in the range 000-999, inclusive. As with the AMC 10 and AMC 12 exams, the AIME competition is offered twice each year. Throughout this book, problems from the first version of a year's exam with be denoted by "AIME" and problems from the second version will be denoted by "AIME-2". Since students mark their answers on a scanned answer sheet, it is imperative that all three digits of the answer are accurately recorded. This includes the possibility of leading zeros in the answer. For instance, the answer 46 should be marked as 046 on the answer sheet. Students have three hours to

[1] There is also an AMC 8 exam given each year in November that is open only to students in eighth grade or younger. That exam is not the subject of this volume.

complete the exam, and no calculators are allowed. The problems appearing on the AIME are noticeably more challenging than those on the AMC 10 and AMC 12 exams overall, but still, they do not require knowledge of calculus. Some of the most common themes in the AIME problems are algebra, combinatorics (i.e., counting), probability, number theory, sequences, logarithmic and exponential functions, trigonometry, complex numbers, polynomials, and geometry. However, any topic at the high school level is fair game, and problems on the AIME often require students to use ideas from multiple mathematical disciplines to obtain the solution.

Students who achieve high scores on both the AMC and AIME may then be selected to compete in the United States of America Mathematical Olympiad (USAMO). Selection criteria are not fixed, but roughly 500 students nationwide are chosen each year. The USAMO is a six-problem essay competition spanning two days. The problems contained in the USAMO are extremely challenging and require great ingenuity and creative thinking. From the USAMO participants, six are chosen to represent the United States at the International Mathematics Olympiad (IMO) each summer. Several texts and resources have been developed on the subject of the USAMO and IMO competitions. The AIME, however, has been comparatively underrepresented in materials and resources for coaches, teachers, and students. The present volume aims to make some headway in this area by providing a resource for teachers and students preparing for the AIME.

An Integer Answer from 000 to 999

We indicated that every answer to every question appearing on the AIME is a three digit integer in the range 000 to 999. This may seem to place severe limitations on the breadth of questions. However, there are several standard ways that questions whose answers do not meet this criterion are typically modified so that the answers do comply. Here are just a few:

- If the actual answer x is a whole number larger than 999, the question can be modified to ask for the leftmost three digits of x or for the remainder when x is divided by 1000.
- For some negative answers x, the question can ask for the absolute value, $|x|$.
- If the actual answer x is a real number in the interval [0, 1000], the question might ask for either the ceiling of x, $\lceil x \rceil$, or the floor of x, $\lfloor x \rfloor$, where

  $$\lceil x \rceil \text{ denotes the smallest integer greater than or equal to } x$$

  and

  $$\lfloor x \rfloor \text{ denotes the largest integer less than or equal to } x.$$

- If the actual answer is a fraction, say $\frac{x}{y}$, the question will usually ask the solver to first simplify the fraction to "lowest terms", $\frac{x}{y} = \frac{m}{n}$, where m and n are relatively prime integers with $n > 0$. This is accomplished by removing common factors in both the numerator x and denominator y. The question can then ask for some integer combination of m and n, such as $m + n$. The subject of relatively prime integers is taken up in Chapter 4

of the book (see Section 4.3). Reducing the fraction is necessary in order to ensure the problem has a unique answer.

- Some AIME questions will ask for the value of m appearing under a radical, such as $\sqrt{m}$. In such cases, it is customary for the question to require us to pull out from under the radical any square factors that occur. For instance, we can rewrite $\sqrt{75}$ as $5\sqrt{3}$. The same applies to any root. For instance, we can rewrite $\sqrt[4]{162} = \sqrt[4]{81 \cdot 2} = 3\sqrt[4]{2}$. In general, by extracting all factors of N of the form p^k (for a prime number p and positive integer $k \geq 2$) from the root of an expression of the form $\sqrt[k]{N}$, we can obtain a unique form for the answer to an AIME problem. Here is an excerpt from an example from Chapter 5: "...can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$." The requirement that m not be divisible by the square of any prime ensures its unique value, as any square factors must be extracted and then simplified with n and p.

  Even though the answer to every AIME problem is an integer in the range 000–999, AIME competitors must be prepared to do hand calculations involving approximations of commonly seen mathematical values. While it is impossible to give a complete list, we briefly enumerate a few of these values here:

x	Approximate Value of x
e	2.71828
π	3.14159
$\ln 2$	0.69315
$\ln 3$	1.09861
$\sqrt{2}$	1.41421
$\sqrt{3}$	1.73205

  The number of decimal places required to obtain a sufficiently accurate approximation depends on the problem, but certainly two or three digits to the right of the decimal point would be an excellent start. This is by no means a complete list, but it is a start and can help to approximate other needed values as well, such as π^2, $\sqrt{6}$, $\ln 8$, and so on.

The Structure of this Book

The first nine chapters in the book are arranged according to mathematical topic, sometimes broken into sections, each of which begins with an introduction that recaps some of the important facts and theorems about that topic that are commonly required to solve AIME problems. The summaries are not intended to serve as a review of everything important about the topic, but focus attention on those aspects that are likely to arise in solving AIME problems. Moreover, in an effort to keep the discourse focused on problem solving, we do not prove rigorously all of the results that are summarized. Of course, understanding why a result is true can often assist the problem solver in applying it effectively, and so far as this can be achieved, some informal discussion of the results will usually be given. Throughout the chapters, some problems that relate to the topic at hand are provided, along with solutions. Most of them are taken directly from past AIMEs. They have been chosen

to display a wide range of strategies, subtopics, and difficulty. It is worth noting that many AIME problems require ideas from multiple branches of mathematics. We have made every effort in this book, however, to place each problem in the chapter of the topic deemed most crucial in understanding the solution. Each chapter concludes with a collection of additional problems on the topic from past AIMEs. A variety have been chosen, generally of gradually increasing difficulty. Readers are encouraged to try to work out solutions to them.

For readers who are not sure how to begin on an exercise, Chapter 10 is a collection of hints that are designed to help the reader get started. It is always more beneficial for the reader to arrive at an independent solution to a problem, rather than reading a published solution. The hints provide the reader an opportunity to enjoy some assistance getting started, but without spoiling the full solution.

The hints provided are divided into two groups. First, each problem has a simple hint designed to get the reader started on one possible track towards a solution. Such hints are usually not too detailed, but provide just enough information to give the reader an idea of where to begin. They appear in the subsection "Hints to Get Started". Then, second, each problem also has a more significant hint (also in Chapter 10) to guide the reader who would benefit from a more comprehensive strategy and plan of solution. They are contained in the subsection "More Extensive Hints". The reader seeking guidance on how to solve a problem in the exercises is encouraged first to consult only the "Hints to Get Started" and then review the "More Extensive Hints" if the first hint is insufficient. In this way, we hope that the book, with its layers of hints and advice, is viewed as indeed being "Gentle", as the title suggests.

Full solutions to all exercises are contained in the final chapter, Chapter 11. Of course, many problems have multiple solutions, and any solution that leads to the correct answer is a good one, even if it is not the one (or, in some cases, two) given in the chapter. Readers should resist the temptation to merely read solutions. For readers seeking to hone problem solving skills, it is better to take hints (Chapter 10) or, if they must, peek at the solution in small doses at a time, all the while trying to engender as much of the solution as they can on their own. Other readers who are taking a more leisurely interest in the material in the book may be content to enjoy the solutions in this manuscript more freely – solutions have been written to be as accessible as possible, often with an eye towards increased accessibility and understandability over and above unlikely or unappreciated cleverness.

Finally, the three digit AIME answer (000 – 999) to every exercise in this book can be found at the back, for readers who simply want to check their answers without even a glance at the hints or solutions.

Advice for Success on the AIME

The casual reader who is using the AIME as a source of fun and recreation can view the present manuscript as a tour guide for a mathematical vacation. For more serious AIME contenders and competitors, however, the road to success requires a lot of practice. That, too, can be found within these pages. While every problem in the book is solved in detail, the AIME student has an opportunity to approach each one as a challenge and as a source of practice for herself or himself.

The practice afforded by this book can be experienced by an individual in isolation if desired, but there are also ways to enhance the experience of problem solving by involving others in the learning process. Math clubs or workshops at school could devote some time to the problems, perhaps turning AIME problems into a little team competition event. Alternatively, teachers could challenge their students with a problem or two out of this book from time to time.

The bottom line is that the AIME problems have the power to change the culture of high school mathematics. AIME preparation doesn't have to take a back seat to the mundane world of high school curriculum standards. There is room for fun and exploration, and some kids are screaming for it, while others might be if only they knew what they are missing. Let us use the AMC program, including the AIME, to ignite a fire within today's young people, old people, and innovative thinkers everywhere who love mathematics or want to grow to love mathematics.

A Word from the Author

If you are reading this book, it means you already have an interest in mathematics competitions, problem solving, creative thinking, and the beauty of mathematics. For that, I applaud you. As a former AIME contestant and current workshop leader and AIME promoter, I can affirm the value of participating in this competition or spending time to ponder the problems. Regardless of your involvement or outcome on the exam, or even if you never enroll as an AIME contestant, the exposure to fun and interesting mathematics, the excitement of discovery, and the time and energy you invest into problems appearing on the AIME all serve to strengthen you as a problem solver, a critical thinker, and a lover of mathematics. The AIME requires you to pursue ideas creatively, and this effort will take you far beyond the normal expectations of the typical high school mathematics classroom. By exploring the problems contained in the AIME, you are honing your mathematical skills for a lifetime, and you are challenging yourself to be the best you can be in mathematics.

In writing this book, I have made every effort to present the mathematics as clearly as possible, in a way that I would teach it to students. I have not always opted to present the shortest, neatest, or most clever solution to a problem. Rather, I have opted always to present a solution that is as accessible as possible, one that a typical student can understand and might say "I could have come up with that". In this way, students will be less likely to be discouraged by a solution that they feel is out of reach and hard to imagine themselves finding on their own.

I have tried to avoid reliance on heavy machinery, deep theorems, or undue abstraction. I have intentionally kept the mathematical proofs in this book in the background, unless they truly reveal insight into how to solve problems and think about the subject matter that they address. Of course, one must come to the AIME with some basic mathematical foundations. Any topics that arise in the AMC 10 and AMC 12 competitions would make a good start. Such common formulas as the quadratic formula or the formulas for factoring a difference of squares or cubes come to mind as examples, but it is hard to give a comprehensive list here. More advanced results that are important in the AIME, such as the Rational Roots Test (Theorem 7.4.6) or the Fundamental Theorem of Arithmetic (Theorem 4.2.1) just to name a couple, are given a bit more introduction and treated with greater detail.

I have written each solution you find in this volume on my own, following my own logic, notations, and ideas. In many cases, I have presented more than one solution to a problem if there are interesting or valuable insights gained. It is possible that the reader may from time to time discover a different solution that they prefer over the ones I presented. That is, of course, perfectly acceptable and expected. It is possible to find other published solutions to the AIME problems in this book either by contacting the American Mathematics Competitions office or via the "Art of Problem Solving" website, www.artofproblemsolving .com.

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