

# Preface

*I begin to understand that while logic is a most excellent guide in governing our reason, it does not, as regards stimulation to discovery, compare with the power of sharp distinction which belongs to geometry.*

— Galileo Galilei (1564–1642)

Geometry combines visual delights and powerful abstractions, concrete intuitions and general theories, historical perspective and contemporary applications, and surprising insights and satisfying certainty. In this textbook, I try to weave together these facets of geometry. I also want to convey the multiple connections that topics in geometry have with each other and that geometry has with other areas of mathematics. The connections link chapters together without sacrificing the survey nature of the whole text.

Geometric thinking fuses reasoning and intuition in a characteristic fashion. The enduring appeal and importance of geometry stem from this synthesis. Mathematical insight is as hard for mathematics students to develop as is the skill of proving theorems. Geometry is an ideal subject for developing both, leading to deeper understanding. However, geometry texts for mathematics majors often emphasize proofs over visualization, whereas some texts for mathematics education majors focus on intuition instead. This book strives to build both and so geometrical thinking throughout the text, the exercises, and the projects. The dynamic software now available provides one valuable way for students to build their intuition and prepare them for proofs. Thus exercises benefiting from technology join hands-on explorations, proofs, and other types of problems. This book builds on the momentum of the NCTM *Standards*, the calculus reform movement, the Common Core State Standards (CCSS), and the ongoing discussion of how to help students internalize mathematical concepts and thinking. (We abbreviate the National Council of Teachers of Mathematics throughout as NCTM.)

There are two natural audiences for a geometry course at the college level—mathematics majors and future secondary mathematics teachers. Of course, the two audiences overlap considerably. In many states, however, requirements for secondary mathematics education majors can differ substantially from those for traditional mathematics majors. This book seeks to serve both audiences, partially by using a survey format so that instructors can choose among topics. In addition, those sections matching CCSS expectations and NCTM recommendations for teacher preparation assume less mathematical sophistication, although they have plenty of problems

and projects to challenge more advanced undergraduates. In particular, in those sections vital for teacher preparation, earlier exercises require less sophistication than later ones. A later subsection of this preface discusses possible course choices.

It has been a treat to revisit the material in this text, re-envision the problems and projects, and add new ones. I have also enjoyed adding the material in Chapters 9 and 10. In the process I realized again how geometric my thinking is and how much I enjoy sharing the elegance and excitement I experience in geometry. I hope that some of my enthusiasm shows through.

## Geometric Intuition

Everyday speech equates intuitive with easy and obvious. However, psychology research confirms what mathematicians have always understood: people build their own intuitions through reflection on their experiences. My students often describe this process as learning to think in a new geometry.

As Galileo's quote introducing this preface suggests, geometry has for centuries been an ideal place for developing mathematical intuition. Since the advent of analytic geometry (at the end of Galileo's life) mathematicians have repeatedly turned geometric insights into algebraic formulations. The applicability, efficiency, simplicity, and power of algebra have reasonably led educators from middle school through graduate school to emphasize algebraic representations. In my view, the success of algebra has so focused the curriculum that students' geometrical thinking often lags far behind. My text tries to correct that imbalance without neglecting the power of algebra. The NCTM calls for high school students to develop geometric intuition and understanding. Similarly the CCSS expects students to build geometrical thinking in a variety of ways.

Throughout this book I seek to help students develop their geometrical intuition. Visualization is an important part of this effort, and the hundreds of figures in the book provide an obvious means to this end. Many of the more than 750 exercises ask students to draw or create their own figures and models. In addition, I have included many exercises and projects requesting students to explore and conjecture new ideas, as well as explain or prove unusual properties. I advocate having students use dynamic geometry software, such as Geometer's Sketchpad or Geogebra, to explore geometric ideas. While some texts explicitly incorporate such software, I don't want to tie my book to one program. However, in my teaching experience students gain different and often more insight working with physical models than from manipulating computer models. So I strongly encourage instructors to give students physical models to use for as many topics as the class time, budget, and their creativity allows. I provide several suggestions in the text and in the projects.

I hope the text's explanations are clear and provide new insights, but I know that students need to reflect on the text and the exercises. I also hope that students find the many non-routine problems and projects challenging, but solvable with effort; such challenges enrich intuition.

## The Role of Proofs

Since Euclid, over two thousand years ago, proofs have had a central place in mathematical thought. Non-mathematicians often think the value of mathematics is restricted to its amazing ability to calculate "answers." Certainly, many applications of mathematics rely on

computational power. However, people applying mathematics want to know more than that there is an answer—after all, an astrologer gives answers. We want to know that the answers are valid. While confidence in much of science depends on experiments, it also often depends on mathematical models. The models make explicit assumptions about how some aspect of the world behaves and recasts them in mathematical terms. So applications of mathematics require that someone—a mathematician—actually prove the results that others use.

However, the need for someone to have proved any given result doesn't lead to a need for every mathematics student to prove every result. Most people acknowledge the value of honing students' ability to reason critically, and mathematical proofs certainly contribute greatly to that skill. Educators actively debate how much students need to prove and at what level of rigor, although all agree that the answers depend on the level of the student. The amount of proof in high school courses now varies greatly across the United States. Still, the NCTM and CCSS call for high school students to do a certain amount of work with proofs. It follows that high school mathematics teachers need substantial background and facility in proof. I have written this text for mathematics majors and future high school mathematics teachers, and I think there is a range of proof experiences both audiences need. All these students need facility in making good arguments in a mathematical context, something I ask for repeatedly throughout the book. At a minimum, that means they need to make their assumptions explicit and use clear reasoning leading from them to their conclusions. I think it also means that they should be introduced to more formal proofs and axiomatic systems, although I don't think that should be the primary focus of an entire course at this level.

In Chapter 1, I employ a fairly informal focus on proof to fit that chapter's goals. One goal is to include enough coverage of the content of Euclidean geometry for students who have not had a solid year-long high school geometry course (and provide a review for others, as needed). Another goal is to develop students' ability to prove substantive results in a context where they already have a comfortable intuition. Hence in that chapter I haven't made explicit the many subtleties and assumptions discussed in Chapter 2. Instead, I ask students to build on Euclid's theorems so that they can prove results that aren't instantly apparent, although they should be plausible. Chapter 2, which looks more carefully at axiomatics, makes axiomatic systems explicit and builds up theorems carefully from the axioms. The level of proof in later chapters lies in between the informality of Chapter 1 and the axiomatically based proofs in Chapter 2, depending on the chapter. In most chapters the content is less familiar and the mathematics more sophisticated than in Chapter 1. Thus the value of proofs in them also connects with the goal of increasing geometric intuition.

Proofs at the college level are written in paragraph form, unlike in high school geometry, where two-column proofs appear frequently. The range of argumentation appropriate at the college level can make the two-column format artificial and overly restrictive. A good proof ought to help the reader understand why the theorem is correct, as well as make the correctness of the reasoning clear, and two-column proofs can sacrifice understanding for clarity of reasoning. Appendix F gives an introduction to proof techniques used in this text.

## General Notation

Proofs end with the symbol ■. Examples end with the symbol ◇. Exercises or parts of exercises with answers or partial answers in the back of the book are marked with an asterisk (\*).

Definitions italicize the word being defined. We use the abbreviation B.C.E. (before the common era) for dates predating the common era, which started somewhat more than 2000 years ago. Dates in the common era will not have the abbreviation C.E. added to them.

## Prerequisites

In general, students need the maturity of Calculus I and II, although only Chapter 9 and Section 2.4 use calculus extensively. Chapter 9 needs some content from Calculus III as well. Appendix E summarizes the material from Calculus III used in Chapter 9. Of course, additional mathematical maturity and familiarity with proofs will help throughout the book. Sections 3.3 and 3.5 and Chapter 9 require an understanding of vectors. Sections 5.3, 5.4, 5.5, 7.3, 7.4, 7.5, 7.6, and 8.4 depend on a more extensive understanding of linear algebra. Appendix D summarizes the linear algebra material needed in the text. Although Chapter 6 builds on concepts from Chapter 5, it doesn't require linear algebra. Section 5.6 makes use of complex numbers and their arithmetic. Chapters 5, 6, and 7 discuss groups and Section 8.4 discusses finite fields, but don't assume any prior familiarity with the concepts.

## Exercises

Learning mathematics centers on doing mathematics, so problems are the heart of any mathematics textbook. A number of exercises appear in the text and are meant to be done while reading that material. Far more appear at the end of each section. I hope that both students and instructors enjoy spending time pondering, solving, discussing, and even extending the problems. They should make lots of diagrams and, when relevant, physical and computer models. The problems include routine and non-routine ones, traditional proofs and computations, hands-on experimentation, conjecturing, and more. Exercises or parts of exercises with answers or partial answers in the back of the book are marked with an asterisk (\*).

## Projects

Too often textbooks and courses shift to a new topic just when students are ready to make their own connections. And geometry is a particularly fertile area for such connections. Projects encourage extending ideas discussed in the text and appear at the end of each chapter. They include essay questions, paper topics, and more extended and open-ended problems. Many of the projects benefit from group efforts. The most succinct projects, of the form "investigate . . .," are leads for paper topics.

## History

Geometry reveals the rich influences over the centuries between areas of mathematics and between mathematics and other fields. Students in geometry, even more than other areas of mathematics, benefit from historical background. The introductory sections of each chapter seek to link the material of the chapter to a broader context and to students' general knowledge. The biographies give additional historical perspective and add a personal flavor to some of the work discussed in the text. One common thread I found in reading about these geometers

was the importance of intuition and visual thinking. As a student I sometimes questioned my mathematical ability because I needed to visualize and construct my own intuitive understanding, instead of grasping abstract ideas directly from a text or a lecture. Now I realize that far greater mathematicians than I built on intuition and visualization for their abstract insights, proofs, and theories. Perhaps this understanding will help the next generation as well.

## Chapter Content

Each chapter starts with an overview, including a discussion of the relevant history, and ends with projects and a list of suggested readings. Geometry is blessed, more than other areas of mathematics, with many wonderful and accessible expository writings as well as texts. (The vast number of web sites, software, and other media devoted to geometry surpass my ability to view, let alone recommend a helpful selection. Further, any printed list would quickly be outdated. So, while I do not make suggestions, I encourage instructors to find media that support their courses.)

1. Euclidean Geometry. Most of this chapter considers plane geometry and follows the lead of the ancient Greeks' approach, especially Euclid's synthesis. (Appendix A gives the definitions, axioms, and propositions of Book I of Euclid's *Elements*.) Since high school geometry courses include much of Euclid's emphases, this approach adds context to a teacher preparation course. Students' preparation varies greatly, so instructors should adjust their pace and coverage according to how much this material is review for the students. All the exercises in Section 1.1 and many of the others have Greek or pre-Greek roots. The three-dimensional material has a more modern focus, considering polyhedra, including geodesic domes, and the sphere. The material on the sphere is a useful transition into a study of non-Euclidean geometry, although it is presented as part of Euclidean geometry.
2. Axiomatic Systems and Models. The first section introduces axiomatic systems and investigates simple ones. The next section considers a high school axiomatic system for Euclidean geometry and Hilbert's axiomatization. (Appendices B and C give the axiomatic systems.) I chose to use the SMSG axioms, one of the "ancestors" of all high school axioms systems, rather than try to choose among contemporary ones. (SMSG is an abbreviation for the School Mathematics Study Group.) The final section explores models and metamathematics. Instructors wishing to include more experience with axiomatic systems and models can include material from Chapter 8.
3. Analytic Geometry. While high school students use analytic geometry, they often don't understand it and often don't see many of the traditional topics. And, although calculus texts include topics such as parametric equations and polar coordinates, instructors often leave them out for lack of time. In addition to these topics, later sections discuss Bézier curves in computer aided design and geometry in three and more dimensions.
4. Non-Euclidean Geometry. The bulk of the chapter develops hyperbolic geometry axiomatically. In addition to typical axioms, we assume the first twenty-eight of Euclid's theorems, which also hold in hyperbolic geometry. By assuming them we can use familiar approaches to focus on how this geometry differs from Euclidean geometry. Models help illustrate the concepts and theorems. The final section considers spherical and single elliptic geometry.

5. **Transformational Geometry.** The first two sections develop the key ideas of transformations and plane isometries without linear algebra. The CCSS strongly emphasize transformations in high school geometry, so at least these first two sections are vital for teacher preparation. Students who have already studied Chapter 4 can consider the corresponding theorems in hyperbolic and spherical geometries. (See Project 22.) The next three sections use linear algebra to delve into isometries more deeply, and into similarities, affine transformations, and transformations in higher dimensions. The final section investigates inversions using complex numbers and relates to the Poincaré disk model of hyperbolic geometry and is not used elsewhere in the text. Appendix D covers the linear algebra needed for this and subsequent chapters.
6. **Symmetry.** While this material uses concepts from Chapter 5, it doesn't depend on linear algebra. Students find this material accessible, compared with some of Chapter 5, and gain insight into the power of the transformational approach, including for applications.
7. **Projective Geometry.** Projective geometry historically and pedagogically provides a capstone unifying Chapters 1, 4, and 5. The first two sections briefly develop it intuitively and axiomatically. Later sections use linear algebra extensively and provide connections with computer graphics and the special theory of relativity.
8. **Finite Geometry.** Since the late nineteenth century geometers have drawn important insights about traditional geometry from the study of simplified finite systems. Section 8.2 discusses the most important of these, finite affine and projective planes, axiomatically. Section 8.3 generalizes the material to balanced incomplete block designs. The final section explores analytic models of finite affine and projective planes and spaces over the fields  $\mathbb{Z}_p$ , the integers (mod  $p$ ), where  $p$  is prime.
9. **Differential Geometry.** Differential geometry deserves an entire undergraduate semester course, but many schools can't offer it. I think students in a survey course benefit from an introduction to this vital area of geometry. I try to convey here its geometric insight and introduce some key geometric ideas—curvature and geodesics, and I endeavor to minimize the machinery of multivariable calculus. The chapter connects differential geometry to Euclidean, spherical, and hyperbolic geometries. Students need little more from multivariable calculus than a familiarity with parametric equations, partial derivatives, and cross products. Appendix E covers the needed multivariable material.
10. **Discrete Geometry.** This relatively new and rapidly growing area focuses on problems, especially ones that remain unsolved. Therefore I organized the chapter around problems. Students are encouraged to explore them in the first section. Subsequent sections develop them more fully, with relevant theorems and more in-depth problems.

## Course Suggestions

This text supports a variety of approaches to geometry and different levels of coverage of the material. Many of the sections benefit from more than one class period, especially to enable students to present problems or projects. The entire book would require a full year geometry course to cover, a luxury few mathematics department can offer.

- A. **Teacher Preparation.** To meet the goals of the CCSS and of the NCTM for teacher preparation, a course should include at least Euclidean geometry (Chapter 1), axiomatic systems

and models (Chapter 2), transformational geometry (Chapter 5, except Section 5.6), and some of non-Euclidean geometry (Chapter 4). As time, interest and student background indicate, topics from analytic geometry (Chapters 3) and symmetry (Chapter 6) are valuable supplements for future teachers.

- B. Historical Survey. Chapters 1, 2, 4, 5, and 7 provide an understanding of the important historical sweep of geometry through the nineteenth and early twentieth centuries. In 1800 there was just Euclidean geometry (Chapter 1). Geometrical thinking expanded enormously, including non-Euclidean geometry (Chapter 4), transformational geometry (Chapter 5), and projective geometry (Chapter 7), among others. The transformations of projective geometry provided a vital unification of geometric thought, both historically and pedagogically. Because of these advances, mathematicians realized the need for a careful investigation of proofs, theories, and models (Chapter 2).
- C. Euclidean Geometry. Chapters 1, 2, 3, 5, and, as time permits, topics from Chapters 6, 9, and 10. If the class needs little review of Euclidean geometry, instructors could interleave Chapters 1 and 10 together at the start of the class.
- D. Transformational Geometry. Chapters 5, 6, and 7 and as much of Chapters 1 and 2 as needed.
- E. Axiomatic Systems and Models. Chapters 1, 2, 4, 8, and Sections 3.1, 7.1, 7.2, 7.3.
- F. Topics. Instructors of courses for mathematics majors have fewer constraints than those teaching mathematics education majors and so can choose topics more freely. Students' background and interest will suggest different options. Students with a weaker background will benefit from Chapters 1, 2, 3, 5, and 6. Chapters 4, 7, 8, 9, and 10 can stretch better prepared students in different ways.

## Dependence and Links Between Chapters

I have tried to keep chapters as independent as reasonable. Students with a decent geometry understanding from high school will have adequate Euclidean and analytic geometry background for all chapters except Chapter 4, which depends explicitly on Chapter 1. The basic concepts of axiomatic systems and models from Chapter 2 appear in Chapters 4, 7, and 8. Chapter 5 is a prerequisite for Chapters 6 and 7. Sections 5.4 and 6.6 consider aspects of fractals. Sections 6.5 and 7.6 briefly consider aspects of the special theory of relativity, and Section 9.4 touches on the general theory of relativity. Section 9.3 refers to Chapter 4.

A number of exercises (denoted with #) connect with other material.

Section 1.1 See Example 1 of Section 10.3 for another proof of Theorem 1.1.2 (the Pythagorean theorem).

# 1.2.9 anticipates Theorem 4.3.1.

# 1.2.10 asks students to prove the converse of the Pythagorean theorem.

# 1.2.14 anticipates Section 4.4.

# 1.2.15 Compare this approach with #3.1.8. Section 10.4 uses this result.

# 1.2.17 is used in a number of later sections.

# 1.2.23, the law of cosines is used in # 3.3.12, # 3.3.21, #3.5.17, #3.5.18, and #10.3.14.

# 1.2.25 is referred to in Section 3.3.

Playfair's axiom appears in Sections 1.3 and 2.2.

# 1.3.8 (d) Compare this approach with # 3.1.4 (a).

Section 1.5 Euler's formula (Theorem 1.5.1) is used in Section 10.4. Project 14 in Chapter 3 asks students to investigate this formula in higher dimensions.

Section 1.5 Shortest paths on a sphere are explored more in # 9.1.6 and in Section 9.4.

Section 1.5 Theorem 1.5.3 relates to Theorem 4.1.1 and their generalization Theorem 9.4.3. # 1.5.35 anticipates concepts in Sections 2.1 and 2.3.

Chapter 1, Project 6 is used in Project 11 of Chapter 10.

Chapter 1, Project 7 relates to the art gallery theorem in Chapter 10.

Many axiomatic systems in Section 2.1 are developed further in Section 2.3. Here are the pairings: Subsection 2.1.3 and # 2.3.6, # 2.1.7 and # 2.3.8, # 2.1.8 and # 2.3.9, # 2.1.9 and # 2.3.11, # 2.1.10 and # 2.3.12, # 2.1.12 and # 2.3.13.

# 2.1.12 and # 2.3.13 anticipate projective planes, discussed in more detail in Section 8.2.

Section 2.3 Example 3 (taxicab geometry) is used in # 3.3.21, in Project 23 of Chapter 5, and in Sections 10.1 and 10.4.

# 3.1.6 and # 3.1.7 relate to Section 2.3.

# 3.1.9 and # 3.1.10 relate to the first fundamental form in Chapter 9.

# 3.1.14 and # 3.1.15 introduce complex numbers and their arithmetic, used in Section 5.6.

Section 3.3 Parametric equations are used extensively in Chapter 9.

# 3.3.17 to # 3.3.22 relate to Section 2.3.

# 3.5.14 connects with Project 2 in Chapter 9.

# 3.5.17 and # 3.5.18 relate to # 10.3.14.

Section 4.1 The pseudosphere is discussed in # 9.3.16.

Section 4.1 Theorem 4.1.1 connects with Theorem 1.5.3 and the generalization Theorem 9.4.3.

Chapter 4, Project 1—Compare with Chapter 5, Project 1 and Chapter 6, Project 3.

# 5.6.10 (b)—Compare with Section 7.1 # 14.

# 5.6.14 relates the Poincaré model and the half plane model of Section 4.1.

Section 6.3 relates to tilings in Sections 10.1 and 10.3.

# 7.5.7 connects with relativistic velocities in Section 6.5 and Lorentz transformations in Section 7.6.

Chapter 7, Project 8 connects Euclidean isometries in Section 5.2 with hyperbolic isometries.

# 8.1.8 investigates an axiomatic system and its models.

## Acknowledgments

This text is based on my book *The Geometric Viewpoint: A Survey of Geometries*, published by Addison Wesley in 1998. First, the acknowledgments from the earlier text: I appreciate the support and helpful suggestions that many people made to improve this book, starting with those students who studied from the rough versions. My previous editors, Marianne Lepp and Jennifer Albanese, and my production supervisor, Kim Ellwood, provided much needed direction, encouragement and suggestions for improvement. I am grateful to the reviewers, whose insightful and careful critiques helped me greatly. They are: Bradford Findell, University of New Hampshire; Yvonne Greenleaf, Rivier College; Daisy McCoy, Lyndon State University; Jeanette Palmiter, Portland State University; and Diana Venters, University of North Carolina,

Charlotte. They pointed out many mistakes and unclear passages; any remaining faults are, naturally, my full responsibility.

I particularly want to thank two people involved in the earlier text. Paul Krueger not only painstakingly made many of the fine illustrations that are an integral part of this book. He also pondered with me over the years the nature of geometric thinking and the connections of geometry with other fields. Connie Gerads Fournelle helped greatly with her valuable perspective on the text, both as a student using an early version and as my assistant writing answers to selected problems.

I want to thank a number of people who have helped in my rewriting this book. First of all, Prof. Peiyi Zhao at St. Cloud State University and Prof. Sue Hagen at Virginia Tech and their students were kind enough to use an early version of this rewrite in their classes and provide me feedback. I thank my students who also used an early version. Prof. Peter Haskell and Prof. Nick Robbins of Virginia Tech kindly provided feedback on Chapter 9. Prof. Ezra Brown, also of Virginia Tech, provided advice for Chapter 8. Prof. Frank Farris of Santa Clara University, pushed me to rewrite this text and provided enthusiastic support over the years. I am grateful to St. John's University for funding the sabbatical making this rewrite possible. Thanks go to the Mathematics Department at Virginia Tech for welcoming me, encouraging me, and providing the materials and space needed in the long rewriting process. I appreciate the extensive and cheerful secretarial support Suzette Ehlinger and Gail Schneider have provided. I am grateful to the editorial staff at the Mathematical Association of America for their hard work: Beverly Ruedi, Carol Baxter, the copy editor, and editors Stan Selzer and Zaven Karien provided the technical support and positive encouragement needed to make this book a reality. Finally I thank my wife, Jennifer Galovich, for her unswerving support and love throughout the years of the writing and now rewriting.

If you have comments or suggestions for improvement, please contact me by e-mail at [tsibley@csbsju.edu](mailto:tsibley@csbsju.edu).

Thomas Q. Sibley  
Mathematics Department  
St. John's University  
Collegeville, MN 56321-3000