Preface

*The Lebesgue Integral for Undergraduates* presents the Lebesgue integral at an understandable level with almost no prerequisites. The text is accessible to anyone who has mastered the single-variable calculus concepts of limits, derivatives, and series. Its material is important to the development of mathematics majors in function theory. These ideas rely on many advanced topics, such as the definition of countable infinity (vs. uncountable infinity), but students should be able to work through the ideas successfully, as they are presented at an introductory level without need for prior exposure. In this way, the mathematics under discussion can be well understood.

This text provides a new ability to learn about Lebesgue integration as a standard course in the undergraduate curriculum. Mathematicians have long understood the course's benefits, since function theory depends on it, but until now have been able to offer it only at the advanced level. For example, it exists at a few institutions as Real Variables II (or a similar variation), and this text works well there. In fact, a student can now take Real Variables I and Real Variables II in either order whenever this text forms the basis of the latter course. Departments no longer need to worry if enough students are coming out of Real Variables I to populate the follow-up course. In addition, the book allows the mathematics undergraduate curriculum to do more. New options exist because the Lebesgue integral now does not depend on complicated measure theory nor a Real Variables I prerequisite. What are some of the results? A Lebesgue Integral course can serve as an elective, similar to Complex Analysis. It can enroll students immediately after Calculus II, or after a first course in mathematical proofs (a transition course), or as a required course in function theory. In fact, students can take the three courses *The Lebesgue Integral, Complex Analysis, and Real Variables* in any order, where each one enhances the other two. Along with Vector Calculus and Probability Theory, these courses allow students to experience a substantially complete undergraduate investigation into functions. *The Lebesgue Integral for Undergraduates* is the tool needed to provide such options.

The text also makes undergraduate research in modern function theory possible. In this way it rejuvenates function theory at a basic undergraduate level. After using this book, undergraduates should have better access to current research questions in the field. They should be able to perform collaborative research. They should be able to appreciate the developments in function theory in the 20th century and beyond, as the Lebesgue integral is the key. This new passage provides a way to lead undergraduates *en masse* to function theory research: institutions that currently do not teach the Lebesgue integral can now offer additional regular
course opportunities or independent studies for students to learn it. The door to function theory research then swings open, and young mathematicians can enter.

The main reason for the textbook’s success at the undergraduate level is its use of a method labeled the “Daniell-Riesz approach,” which is briefly explained in the Introduction and then presented in Chapter 1. This text’s main goal is to present Lebesgue’s integral in an understandable way, including integrals defined in terms of general Borel measures. We want readers to get a real “feel” for what is involved with the integral, and so most of the exercises are calculational. Theoretical ideas and issues are also presented, but several concepts are intentionally omitted. For example, the equivalence of the $L^1$ spaces obtained by Lebesgue’s method vs. the Daniell-Riesz approach is important and is verified in other texts (cf. [121, pp. 127–130] and [98, p. 96]), but it is not this book’s focus. The text’s expectation is thereby restricted but holistic: students gain a complete insight in how the integral is defined and works in fundamental ways, which is different from the traditional main goal of a graduate text on the subject (where students prove advanced theoretical results). Both goals can be accomplished. In fact, this text has both theoretical presentations and exercises for the students to explore theoretical concepts, but they are chosen carefully so as not to overwhelm the student or expand the size of a manageable text. Most theoretical exercises investigate a straightforward property or practice using simple theoretical topics (such as a definition) in a proof setting. A corresponding learning objective is for undergraduate readers (even at an introductory level) to gain confidence and practice with manageable and straightforward proofs.

This book uses the Daniell-Riesz approach throughout, but presents it in a way that makes the material understandable to almost a beginning undergraduate mathematician. The joys of this study are numerous. Here are just three:

The mathematics is fun. Presented in as simple a format as possible, it is often surprising. For example, there are two infinities (as sizes of sets of real numbers); in fact, there are at least two. There are functions with only two output values that cannot be integrated—at least using Riemann sums. There are spaces of functions with a geometry that matches the Euclidean geometry for spaces of points.

It is important. Lebesgue’s integral opens up modern function theory to the student. His integral is required to understand the current research of function theorists. It provides a gateway into the modern mathematics of functions, both real and complex.

It supports ideas learned in other mathematics courses that describe functions, such as real analysis and complex analysis. Learned before a student takes these courses (or at the same time, or afterwards), Lebesgue’s integral becomes another “card in the student’s deck” to see what functions are all about.

The last goal for the text is to discuss applications. They are advanced ideas in function theory, but the book does its best to present them in a manageable package—in an introductory fashion that works for undergraduates. Other books (cf. [86]) do a wonderful job of laying out a complete theoretical framework (and giving a full treatment of associated applications), but that task requires a lengthy development impossible here. Such thoroughness is not the goal of this text. Instead, The Lebesgue Integral for Undergraduates develops the integral’s definition, describes its characteristics useful in other contexts within the text, and provides a small taste of many function theoretic topics that evolve out of the integral (though these are mainly limited
to real functions). Other mathematicians might choose a different set of topics with persuasive reasons. For example, it is tempting (as an expansion that would serve many students well) to have included more material on complex-valued functions. Or a more general discussion of Banach spaces. Or $C^*$-algebras. And so on. The topics that made the book’s “final cut,” such as Hilbert space, are not all encompassing—that was impossible for this project—but they are selectively impressive and enjoyable to teach and learn. Follow-up investigations can always take place. In a forward-looking way, the text concludes by inviting undergraduate students into further study, including the traditional measure-theoretic graduate Lebesgue course. They can see the broad plain of function theory stretching before them, and they can choose to step forth in exploration.

This book explains all the necessary real analysis concepts, such as limits of sequences, continuity, and a set’s supremum and infimum values. These prerequisites are grounded in discussions that have clear and interesting payoffs: they must be used in order to define the Lebesgue integral and its applications. Finally, the material is modern, which motivates its study as relevant—current function theory research uses it.

After extensive classroom use and student feedback, the text includes many features to enable learning. Each section contains embedded Questions; they help create a “workbook” aspect and make sure ideas are understood. Answers appear at the end of each section. The text also has Reading Questions (a quick sectional review check), about 700 exercises (for practice on both calculational problems and proof writing), solutions to odd-numbered exercises, historical vignettes and Chapter Notes (that include stories of many great mathematicians, celebrating the diversity of mathematical talent), and an application at the end of each chapter (describing an idea with powerful implications—for our physical world, for the way we think about chance, or for an advanced mathematical topic that intrigues us).

Special acknowledgment and thanks go to Stanley Seltzer, Steven Kennedy, Carol Baxter, and many others at the MAA for their supportive professional work to bring this textbook to publication. The (anonymous) reviewers’ comments and the copy editor were particularly helpful and dramatically improved the exposition. Many friends, colleagues, and students supported this project. Jay Howard at Butler University provided funding from his office. Others who nourished it at critical stages include Scott Chapman, Carl Cowen, Patricia Johnston, Susan Johnston, John Mugge, James Rovnyak, Steven Seubert, Derek Thompson, John Wilson, Phyllis Cohen, Jeremiah Farrell, and André Wehner. I thank them. Freshmen to seniors at three academic institutions have taken the integration course and have given feedback, improving the explanations considerably. My success at teaching and writing about the Lebesgue integral has occurred because of this support. Any error in this book is a result of my work and is my responsibility, but describing the Lebesgue integral has provided great joy for which I am profoundly grateful.