Introduction

We are deeply saddened by the passing of **Martin Gardner** on May 22, 2010. It is the end of an era in the world of popular and recreational mathematics. Thus this book is dedicated to his memory.

Martin's legacy in popular and recreational mathematics is legendary, though he modestly insisted that he was not a mathematician but a mathematics journalist. His background was in literature and philosophy, and his most famous book is *The Annotated Alice*. In honor of Martin, we invited the denizens of Lewis Carroll's *Alice in Wonderland* and its companion volume *Through the Looking Glass* to grace the pages of this volume and guide us through our mathematical adventures.

We strongly believe that the primary reason many students are having difficulty with mathematics is that they are bothered by the language of mathematics. By having Alice and the twins, Tweedledum and Tweedledee, serve as the protagonists in this volume, it is hoped that the students may find the conversational style less daunting to read than formal mathematics.

Alice and the twins are featured prominently in Chapter Zero, which is intended as assigned reading. They also introduce us to all subsequent chapters. As the students become more comfortable with the subject, the conversations recede to the background so as not to be overly distracting.

We have taken many excerpts from both books, but also bent them to our purpose. We hope Lewis Carroll would pardon the liberty we had taken, especially in mixing the characters between the two books. We wish to arouse sufficient interest in the students so that they would actually read and enjoy these two books, other works of Lewis Carroll, and classical literature in general.

At the University of Alberta, this book is used as a text for a contents course, offered by the Faculty of Science, for students in the elementary education program. These students also take Curriculum and Instruction courses from the Faculty of Education. Although the book has these students in mind all the time, it does not offer much pedagogical advice. We leave this to the experts.

Even as a text for a contents course, the book is still somewhat idiosyncratic and unorthodox. Many instructors may see it more as a classroom resource. Also, because its entry point is set at the ground level, the book is very suitable for self-study, life-long learning as well as liberal arts education.

It should be emphasized that we are not trying to teach mathematics in the elementary school classroom, but what lies behind the subject. It is impossible to satisfy everyone, but much thought has been devoted to how we could serve both average students and highly motivated and talented students.

In the selection of material to be covered, we have restricted our attention to arithmetic only. This affords an in-depth study, as opposed to a broader coverage that would necessarily be more superficial. Nevertheless, even within the confines of arithmetic, we have covered a lot of ground. Thus there are plenty of options in steering through the book.

We have included a lot of proofs, so that our treatment of the subject is on solid ground. Some of the arguments are given in everyday language. For instance, the introduction to the Euclidean Algorithm is via selling apples and bananas, and its justification is via the comparison of two photographs. How much proof should be covered is at the discretion of the instructor.

A cursory scan of the Table of Contents reveals that this book consists of eight chapters, each divided into four sections. The last section of each chapter is titled Extras and consists of related optional material, such as the Cancellation Law for the multiplication of congruences in Section 3.4. With this preamble, we now embark on a quick tour of the text.

Chapter Zero on the Review of Arithmetic may be assigned as reading, or taken up in class at the beginning of the course. It is there mostly for referencing. Its writing is influenced by the classic *A Survey of Modern Algebra* by **Garrett Birkoff** and **Saunders MacLane**. Much of the same ground is covered in greater details in Chapter Seven.

Chapter One on Divisibility is in some sense the soul of this book. The overall discussion revolves about the concept of division, and this chapter deals with exact division. It contains a most important result, the Basic Divisibility Theorem, and introduces the standard symbol for divisibility.

Chapter Two on Congruence deals with inexact division in one way. Traditionally, this topic comes up much later, largely because of the introduction of a new concept and a new symbol not often encountered in the school curriculum. Our emphasis is on the relation between divisibility and congruence. However, we take the position that it can stand on its own as a topic for early study.

Chapter Three on Common Divisors and Multiples is the heart of this book. Although we are in general against the proliferation of symbols, we introduce two here that we feel are long overdue. They are $\triangle$ and $\bigtriangledown$, denoting the greatest common divisor and least common multiple respectively. Our notations emphasize the fact that they are operations, and the symbolism is consistent with the logic connectives “and” and “or” that underlie these two concepts.

Chapter Four on Diophantine Equations is essentially an extension of the preceding chapter. Here students will encounter problems that require a heavy dose of computations. Many students find this rough going, but once they catch on to the idea that algorithms are mechanical processes that can be learnt methodically, they feel empowered that they can actually carry out such seemingly complicated calculations. Section 4 is definitely for the students looking for more.

Chapter Five on Prime Factorizations perhaps comes much later than in most textbooks. In this we followed the example of the illustrious **Euclid**, who delayed the discussion of parallelism until the latest possible moment in his monumental treatise, *Elements of Geometry*, even though the concept of parallelism is defined much earlier. Our feeling is that although we can define prime numbers much earlier on, problems related to prime numbers are on the whole the most difficult. The Fundamental Theorem of Arithmetic is a powerful tool, but its functioning often depends on finding prime factorizations of numbers, which is a non-trivial problem.

Chapter Six on Rational and Irrational Numbers deals with inexact divisions in another way. Up to this point, we work exclusively with integers. Here we introduce the concept of fractions and decimals. Many elementary school mathematics teachers are hazy about them, and this chapter guides them through painstakingly.

Chapter Seven on Numeration Systems returns to the integers under assumed names. In many ways, this is the easiest of all the chapters. It is not a bad idea to cover this topic last, as we tend to run out of time towards the end of a course. We deliberately deemphasize conversions and focus on immersion in other basis. We are not trying to switch to other numeration systems, but use them to reinforce better understanding of our own.

Each section has 8 examples and 12 exercises. The end of an example is marked by a row of asterisks. The exercises vary from the routine to the very challenging. Thus there is much material to choose from when making up assignments. A manual containing the solutions to even-numbered exercises is available from the publishers.

We have deliberately kept the total number of Theorems to a minimum. We do not call a result a Theorem unless we have used it on at least one occasion, preferably more. We also give short names to each Theorem to make it easier for references, while giving a hint of its content. This is of course another instance of our imitating Euclid.

We have decided not to include a bibliography section, as nowadays most people look things up on the internet. Our book is reasonably self-contained. Upon a more thorough reading, two names will emerge in the readers' mind. Apart from the work of Lewis Carroll and Martin, we should mention **Raymond Smullyan's** fantastic *Alice in Puzzleland*.

I am grateful to **Martin Gardner** for his influence on my life and career, for being a regular correspondent and personal friend over the years, and for writing a Preface to a preliminary edition of this book. My friends and colleagues **Sean Graves, Wanida Hemakul, Cynthia Huffman, Charles Leytem, Trevor Pasanen, Wen-Hsien Sun** and **Paul Vaderlind** have all read the manuscript, pointed out numerous typos and made valuable suggestions.

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