About these Study Guides

The Mathematical Association of America's American Mathematics Competitions' website, www.maa.org/math-competitions, announces loud and clear:

> Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

For over six decades dedicated and clever folk of the MAA have been creating and collating marvelous, stand-alone mathematical tidbits and sharing them with the world of students and teachers through mathematics competitions. Each question devised serves as a portal for deep mathematical mulling and exploration. Each is an invitation to revel in the mathematical experience.

And more! In bringing together all the questions that link to one topic a coherent mathematical landscape, ripe for a guided journey of study, emerges. The goal of this series is to showcase the landscapes that lie within the MAA's competition resources and to invite students, teachers, and all life-long learners, to engage in the mathematical explorations they invite. Learners will not only deepen their understanding of curriculum topics, but also gain the confidence to play with ideas and work to become agile intellectual thinkers.

I was recently asked by some fellow mathematics educators what my greatest wish is for our next generation of students. I responded:

> . . . a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try some more, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas," revel in success, and yearn for more.

Our complex society demands of our next generation not only mastery of quantitative skills, but also the confidence to ask new questions, to innovate, and to succeed. Innovation comes only from bending and pushing ideas and being willing to flail. One must rely on one's wits and on one's common sense. And one must persevere. Relying on memorized answers to previously asked—and answered!—questions does not push the frontiers of business research and science research.

The MAA competition resources provide today's mathematics thinkers, teachers, and doers:

- the opportunity to learn and to teach problem-solving, and
- the opportunity to review the curriculum from the perspective of understanding and clever thinking, letting go of memorization and rote doing.

Each of these study guides

- runs through the entire standard curriculum content of a particular mathematics topic from a sophisticated and mathematically honest point of view,
- illustrates in concrete ways how to implement problem-solving strategies for problems related to the particular mathematics topic, and
- provides a slew of practice problems from the MAA competition resources along with their solutions.

As such, these guides invite you to

- review and deeply understand mathematics topics,
- practice problem-solving,
- gain incredible intellectual confidence,

and, above all,

- to enjoy mathematics!

This Guide and Mathematics Competitions

Whether you enjoy the competition experience and are motivated by it and delighted by it, or you, like me, tend to shy away from it, this guide is for you!

We all have our different styles and proclivities for mathematics thinking, doing, and sharing, and they are all good. The point, in the end, lies with the enjoyment of the mathematics itself. Whether you like to solve problems under the time pressure of a clock or while mulling on a stroll, problem-solving is a valuable art that will serve you well in all aspects of life.

This guide teaches how to think about content and how to solve challenges. It serves both the competition doers and the competition non-doers. That is, it serves the budding and growing mathematicians we all are.

On Competition Names

This guide pulls together problems from the history of the MAA's American competition resources.

The competitions began in 1950 with the Metropolitan New York Section of the MAA offering a "Mathematical Contest" each year for regional high-school students. They became national endeavors in 1957 and adopted the name "Annual High School Mathematics Examination" in 1959. This was changed to the "American High School Mathematics Examination" in 1983.

In this guide, the code "#22, AHSME, 1972," for example, refers to problem number 22 from the 1972 AHSME, Annual/American High School Mathematics Examination.

In 1985 a contest for middle school students was created, the "American Junior High School Mathematics Examination," and shortly thereafter the contests collectively became known as the "American Mathematics Competitions," the AMC for short. In the year 2000 competitions limited to high-school students in grades 10 and below were created and the different levels of competitions were renamed the AMC 8, the AMC 10, and the AMC 12.

In this guide, "#13, AMC 12, 2000," for instance, refers to problem number 13 from the 2000 AMC 12 examination.

In 2002, and ever since, two versions of the AMC 10 and the AMC 12 are administered, about two weeks apart, and these are referred to as the AMC 10A, AMC 10B, AMC 12A, and AMC 12B.

In this guide, "#24, AMC 10A, 2013," for instance, refers to problem number 24 from the 2013 AMC 10A examination.

On Competition Success

Let's be clear:

> "I am using this guide for competition practice. Does this guide promise me 100% success on all mathematics competitions, each and every time?"
>
> Of course not!

But this guide does offer, if worked through with care

- Feelings of increased confidence when taking part in competitions.
- Clear improvement on how you might handle competition problems.
- Clear improvement on how you might handle your emotional reactions to particularly outlandish-looking competition problems.

Mathematics is an intensely human enterprise and one cannot underestimate the effect of emotions when doing mathematics and attempting to solve challenges. This guide gives the human story that lies behind the mathematics content and discusses the human reactions to problem-solving.

As we shall learn, the first and the most important, effective step in solving a posed problem is

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This guide provides practical content knowledge, problem-solving solving tools and techniques, and concrete discussion on getting over the barriers of emotional blocks. Even though its goal is not necessarily to improve competition scores, these are the tools that nonetheless lead to that outcome!

This Guide and the Craft of Solving Problems

Success in mathematics—however you wish to define it—comes from a strong sense of self-confidence: the confidence to acknowledge one's emotions and to calm them down, the confidence to pause over ideas and come to educated guesses or conclusions, the confidence to rely on one's wits to navigate through unfamiliar terrain, the confidence to choose understanding over impulsive rote doing, and the confidence to persevere.

Success and joy in science, business, and in life doesn't come from programmed responses to pre-set situations. It comes from agile and adaptive thinking coupled with reflection, assessment, and further adaptation.

Students—and adults too—are often under the impression that one should simply be able to leap into a mathematics challenge and make instant progress of some kind. This not how mathematics works! It is okay to fumble, and flail, and to try out ideas that turn out not help in the end. In fact, this *is* the problem-solving process and making multiple false starts should not at all be dismissed! (Think of how we solve problems in everyday life.)

It is also a natural part of the problem-solving process to react to a problem.

"This looks scary."
"This looks fun."
"I don't have a clue what the question is even asking!"
"Wow. Weird! Could that really be true?"
"Who cares?"
"I don't get it."
"Is this too easy? I am suspicious."

We are each human, and the first step to solving a problem is to come to terms with our emotional reaction to it—especially if that reaction is one of being overwhelmed. Step 1 to problem-solving mentioned in the previous section in vital.

Once we have nerves in check, at least to some degree, there are a number of techniques one could try in order to make some progress with the problem.

The ten strategies we briefly outline in the appendix are discussed in full detail on the MAA's CURRICULUM INSPIRATIONS webpage, www.maa.org/ci. There you will find essays and videos explaining each technique in full, with worked examples and slews of further practice examples and their solutions.

This guide also contains worked examples. Look for the FEATURED PROBLEMS in sections 1, 3, 5, 6, 10, 12, 13, 14, 16, 17, and 18 where I share with you my own personal thoughts, emotions, and eventual approach in solving a given problem using one of the ten problem-solving strategies.

This Guide and Mathematics Content: Trigonometry

This guide covers the story of trigonometry. It is a swift overview, but it is complete in the context of the content discussed in beginning and advanced high-school courses. The purpose of these notes is to supplement and put into perspective the material of any course on the subject you may have taken or are currently taking. (These notes will be tough going for those encountering trigonometry for the very first time!)

In reading and working through the material presented here you will

- see the story in of trigonometry in a new light,
- see the reasons why we, mankind, developed the subject in the way we did,
- begin to move away from memorization and half-understanding to deep understanding, and thereby
- be equipped for agile, clever thinking in the subject.

These notes will guide you through to sound mathematical doing in trigonometry and, of course, to sound problem-solving skills as well.

For Educators: This Guide and the Common Core State Standards

The very first Standard for Mathematical Practice asks—requires!—that we educators pay explicit attention to teaching problem-solving:

MP1 Make sense of problems and persevere in solving them.

And one can argue that several, if not all, of the remaining seven Standards for Mathematical Practice can play prominent roles in supporting this first standard. For example, when solving a problem, students will likely be engaging in the activities of these standards too:

MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

These guides on *Clever Studying through the MAA AMC* align directly with the Standards for Mathematical Practice.

And each individual guide directly addresses content standards too! This volume on trigonometry attends to the following standards. (The section numbers refer to the sections of this text in which the standards appear.)

8.G.6 Explain a proof of the Pythagorean Theorem and its converse. (Sections 1 and 14.)

8.G.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. (Section 1.)

8.G.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system. (Section 1.)

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (Section 4.)

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (Sections 2, 3, 4, and 11.)

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sines, cosines, and tangents for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. (Sections 3 and 7.)

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. (Sections 3, 5, 6, and 7.)

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. (Section 18.)

F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. (Section 12.)

F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. (Section 12.)

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. (Section 7.)

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. (Section 13.)

G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (Section 10.)

G-SRT.7 Explain and use the relationship between the sine and cosine of complementary angles. (Section 10.)

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (Sections 8, 9, and 10.)

G-SRT.9 (+) Derive the formula $A = 1/2\, ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. (Sections 15 and 17.)

G-SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems. (Sections 14, 16, and 17.)

G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces). (Sections 14 and 16.)