CHAPTER 2

Circles

Construct a circle of radius zero. . .

Although it is often an intermediate step, angle chasing is usually not enough to solve a problem completely. In this chapter, we develop some other fundamental tools involving circles.

2.1 Orientations of Similar Triangles

You probably already know the similarity criterion for triangles. Similar triangles are useful because they let us convert angle information into lengths. This leads to the power of a point theorem, arguably the most common sets of similar triangles.

In preparation for the upcoming section, we develop the notion of similar triangles that are similarly oriented and oppositely oriented.

Here is how it works. Consider triangles ABC and XYZ. We say they are **directly similar**, or similar and **similarly oriented**, if

$$\measuredangle ABC = \measuredangle XYZ, \ \measuredangle BCA = \measuredangle YZX, \ \text{and} \ \measuredangle CAB = \measuredangle ZXY.$$

We say they are **oppositely similar**, or similar and **oppositely oriented**, if

$$\measuredangle ABC = -\measuredangle XYZ, \ \measuredangle BCA = -\measuredangle YZX, \ \text{and} \ \measuredangle CAB = -\measuredangle ZXY.$$

If they are either directly similar or oppositely similar, then they are **similar**. We write $\triangle ABC \sim \triangle XYZ$ in this case. See Figure 2.1A for an illustration.

Two of the angle equalities imply the third, so this is essentially directed AA. Remember to pay attention to the order of the points.

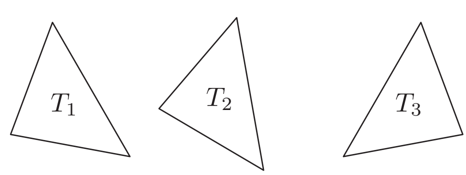

Figure 2.1A. T_1 is directly similar to T_2 and oppositely to T_3.

The upshot of this is that we may continue to use directed angles when proving triangles are similar; we just need to be a little more careful. In any case, as you probably already know, similar triangles also produce ratios of lengths.

Proposition 2.1 (Similar Triangles). *The following are equivalent for triangles ABC and XYZ.*

(i) $\triangle ABC \sim \triangle XYZ$.
(ii) (AA) $\angle A = \angle X$ *and* $\angle B = \angle Y$.
(iii) (SAS) $\angle B = \angle Y$, *and* $AB : XY = BC : YZ$.
(iv) (SSS) $AB : XY = BC : YZ = CA : ZX$.

Thus, lengths (particularly their ratios) can induce similar triangles and vice versa. It is important to notice that SAS similarity does not have a directed form; see Problem 2.2. In the context of angle chasing, we are interested in showing that two triangles are similar using directed AA, and then using the resulting length information to finish the problem. The power of a point theorem in the next section is perhaps the greatest demonstration. However, we remind the reader that angle chasing is only a small part of olympiad geometry, and not to overuse it.

Problem for this Section

Problem 2.2. Find an example of two triangles ABC and XYZ such that $AB : XY = BC : YZ$, $\measuredangle BCA = \measuredangle YZX$, but $\triangle ABC$ and $\triangle XYZ$ are not similar.

2.2 Power of a Point

Cyclic quadrilaterals have many equal angles, so it should come as no surprise that we should be able to find some similar triangles. Let us see what length relations we can deduce.

Consider four points A, B, X, Y lying on a circle. Let line AB and line XY intersect at P. See Figure 2.2A.

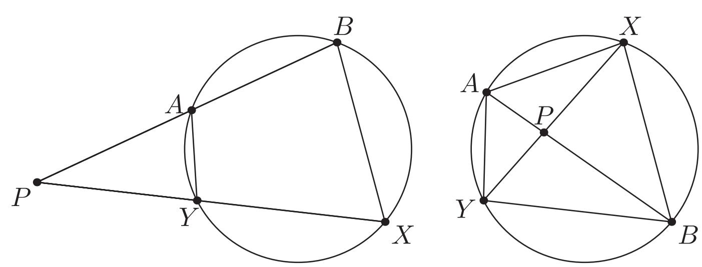

Figure 2.2A. Configurations in power of a point.

A simple directed angle chase gives that

$$\measuredangle PAY = \measuredangle BAY = \measuredangle BXY = \measuredangle BXP = -\measuredangle PXB$$

and

$$\measuredangle AYP = \measuredangle AYX = \measuredangle ABX = \measuredangle PBX = -\measuredangle XBP.$$

As a result, we deduce that $\triangle PAY$ is oppositely similar to $\triangle PXB$.

Therefore, we derive

$$\frac{PA}{PY} = \frac{PX}{PB}$$

or

$$PA \cdot PB = PX \cdot PY.$$

This is the heart of the theorem. Another way to think of this is that the quantity $PA \cdot PB$ does not depend on the choice of line AB, but instead only on the point P. In particular, if we choose line AB to pass through the center of the circle, we obtain that $PA \cdot PB = |PO - r||PO + r|$ where O and r are the center and radius of ω, respectively. In light of this, we define the **power of P** with respect to the circle ω by

$$\operatorname{Pow}_\omega(P) = OP^2 - r^2.$$

This quantity may be negative. Actually, the sign allows us to detect whether P lies inside the circle or not. With this definition we obtain the following properties.

Theorem 2.3 (Power of a Point). *Consider a circle ω and an arbitrary point P.*

(a) The quantity $\operatorname{Pow}_\omega(P)$ is positive, zero, or negative according to whether P is outside, on, or inside ω, respectively.
(b) If ℓ is a line through P intersecting ω at two distinct points X and Y, then

$$PX \cdot PY = |\operatorname{Pow}_\omega(P)|.$$

(c) If P is outside ω and $\overline{PA}$ is a tangent to ω at a point A on ω, then

$$PA^2 = \operatorname{Pow}_\omega(P).$$

Perhaps even more important is the converse of the power of a point, which allows us to find cyclic quadrilaterals based on length. Here it is.

Theorem 2.4 (Converse of the Power of a Point). *Let A, B, X, Y be four distinct points in the plane and let lines AB and XY intersect at P. Suppose that either P lies in both of the segments $\overline{AB}$ and $\overline{XY}$, or in neither segment. If $PA \cdot PB = PX \cdot PY$, then A, B, X, Y are concyclic.*

Proof. The proof is by phantom points (see Example 1.32, say). Let line XP meet (ABX) at Y'. Then A, B, X, Y' are concyclic. Therefore, by power of a point, $PA \cdot PB = PX \cdot PY'$. Yet we are given $PA \cdot PB = PX \cdot PY$. This implies $PY = PY'$.

We are not quite done! We would like that $Y = Y'$, but $PY = PY'$ is not quite enough. See Figure 2.2B. It is possible that Y and Y' are reflections across point P.

Fortunately, the final condition now comes in. Assume for the sake of contradiction that $Y \neq Y'$; then Y and Y' are reflections across P. The fact that A, B, X, Y' are concyclic implies that P lies in both or neither of $\overline{AB}$ and $\overline{XY'}$. Either way, this changes if we consider $\overline{AB}$ and $\overline{XY}$. This violates the second hypothesis of the theorem, contradiction. □

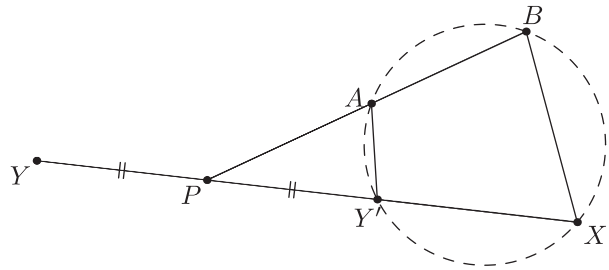

Figure 2.2B. It's a trap! $PA \cdot PB = PX \cdot PY$ almost implies concyclic, but not quite.

As you might guess, the above theorem often provides a bridge between angle chasing and lengths. In fact, it can appear in even more unexpected ways. See the next section.

Problems for this Section

Problem 2.5. Prove Theorem 2.3.

Problem 2.6. Let ABC be a right triangle with $\angle ACB = 90^\circ$. Give a proof of the Pythagorean theorem using Figure 2.2C. (Make sure to avoid a circular proof.)

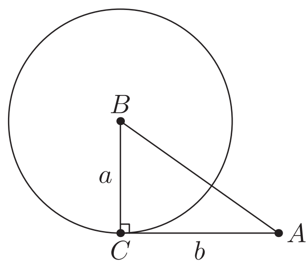

Figure 2.2C. A proof of the Pythagorean theorem.

2.3 The Radical Axis and Radical Center

We start this section with a teaser.

Example 2.7. Three circles intersect as in Figure 2.3A. Prove that the common chords are concurrent.

This seems totally beyond the reach of angle chasing, and indeed it is. The key to unlocking this is the radical axis.

Given two circles ω_1 and ω_2 with distinct centers, the **radical axis** of the circles is the set of points P such that

$$\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P).$$

At first, this seems completely arbitrary. What could possibly be interesting about having equal power to two circles? Surprisingly, the situation is almost the opposite.

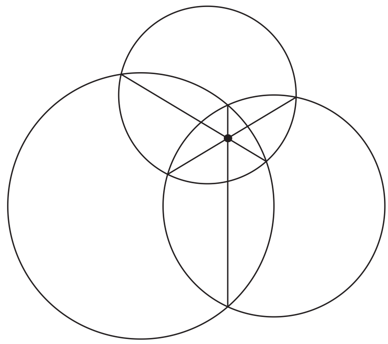

Figure 2.3A. The common chords are concurrent.

Theorem 2.8 (Radical Axis). *Let ω_1 and ω_2 be circles with distinct centers O_1 and O_2. The radical axis of ω_1 and ω_2 is a straight line perpendicular to $\overline{O_1O_2}$.*

In particular, if ω_1 and ω_2 intersect at two points A and B, then the radical axis is line AB.

An illustration is in Figure 2.3B.

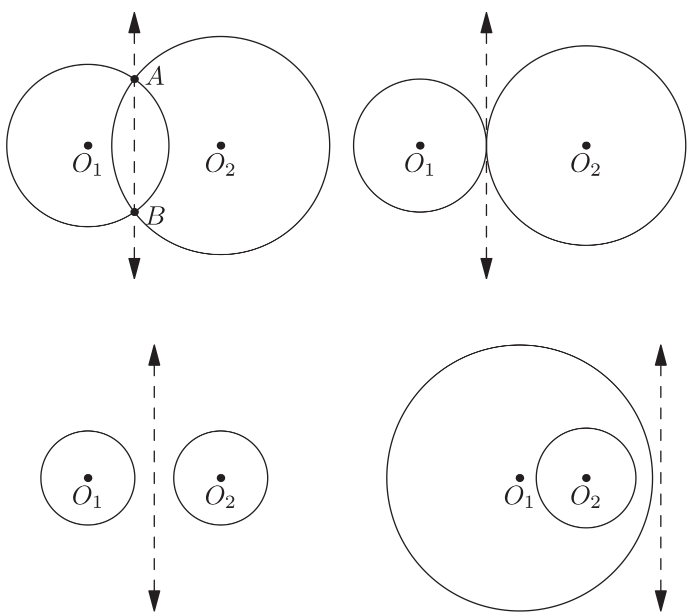

Figure 2.3B. Radical axes on display.

Proof. This is one of the nicer applications of Cartesian coordinates—we are motivated to do so by the squares of lengths appearing, and the perpendicularity of the lines. Suppose that $O_1 = (a, 0)$ and $O_2 = (b, 0)$ in the coordinate plane and the circles have radii r_1 and r_2 respectively. Then for any point $P = (x, y)$ we have

$$\text{Pow}_{\omega_1}(P) = O_1P^2 - r_1^2 = (x-a)^2 + y^2 - r_1^2.$$

Similarly,

$$\text{Pow}_{\omega_2}(P) = O_2P^2 - r_2^2 = (x-b)^2 + y^2 - r_2^2.$$

Equating the two, we find the radical axis of ω_1 and ω_2 is the set of points $P = (x, y)$ satisfying

$$\begin{aligned} 0 &= \text{Pow}_{\omega_1}(P) - \text{Pow}_{\omega_2}(P) \\ &= \left[(x-a)^2 + y^2 - r_1^2\right] - \left[(x-b)^2 + y^2 - r_2^2\right] \\ &= (-2a+2b)x + \left(a^2 - b^2 + r_2^2 - r_1^2\right) \end{aligned}$$

which is a straight line perpendicular to the x-axis (as $-2a + 2b \neq 0$). This implies the result.

The second part is an immediately corollary. The points A and B have equal power (namely zero) to both circles; therefore, both A and B lie on the radical axis. Consequently, the radical axis must be the line AB itself. □

As a side remark, you might have realized in the proof that the standard equation of a circle $(x-m)^2 + (y-n)^2 - r^2 = 0$ is actually just the expansion of $\text{Pow}_\omega((x, y)) = 0$. That is, the expression $(x-m)^2 + (y-n)^2 - r^2$ actually yields the power of the point (x, y) in Cartesian coordinates to the circle centered at (m, n) with radius r.

The power of Theorem 2.8 (no pun intended) is the fact that it is essentially an "if and only if" statement. That is, a point has equal power to both circles if and only if it lies on the radical axis, which we know much about.

Let us now return to the problem we saw at the beginning of this section. Some of you may already be able to guess the ending.

Proof of Example 2.7. The common chords are radical axes. Let ℓ_{12} be the radical axis of ω_1 and ω_2, and let ℓ_{23} be the radical axis of ω_2 and ω_3.

Let P be the intersection of these two lines. Then

$$P \in \ell_{12} \Rightarrow \text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P)$$

and

$$P \in \ell_{23} \Rightarrow \text{Pow}_{\omega_2}(P) = \text{Pow}_{\omega_3}(P)$$

which implies $\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_3}(P)$. Hence $P \in \ell_{31}$ and accordingly we discover that all three lines pass through P. □

In general, consider three circles with distinct centers O_1, O_2, O_3. In light of the discussion above, there are two possibilities.

1. Usually, the pairwise radical axes concur at a single point K. In that case, we call K the **radical center** of the three circles.
2. Occasionally, the three radical axes will be pairwise parallel (or even the same line). Because the radical axis of two circles is perpendicular to the line joining its centers, this (annoying) case can only occur if O_1, O_2, O_3 are collinear.

It is easy to see that these are the only possibilities; whenever two radical axes intersect, then the third one must pass through their intersection point.

We should also recognize that the converse to Example 2.7 is also true. Here is the full configuration.

Theorem 2.9 (Radical Center of Intersecting Circles). *Let ω_1 and ω_2 be two circles with centers O_1 and O_2. Select points A and B on ω_1 and points C and D on ω_2. Then the following are equivalent:*

(a) A, B, C, D lie on a circle with center O_3 not on line O_1O_2.
(b) Lines AB and CD intersect on the radical axis of ω_1 and ω_2.

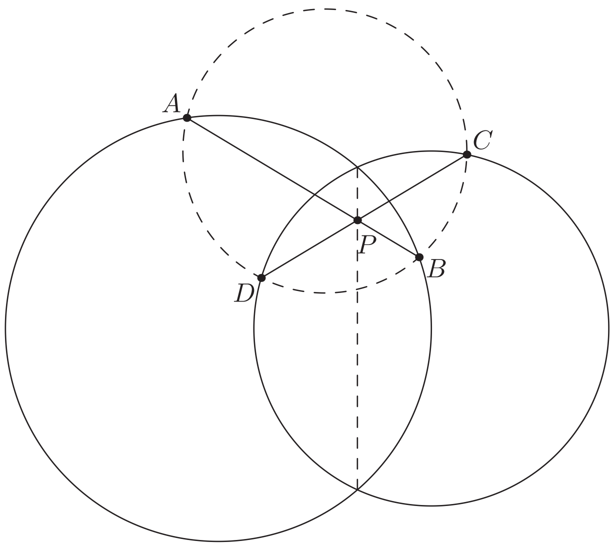

Figure 2.3C. The converse is also true. See Theorem 2.9.

Proof. We have already shown one direction. Now suppose lines AB and CD intersect at P, and that P lies on the radical axis. Then

$$\pm PA \cdot PB = \operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P) = \pm PC \cdot PD.$$

We need one final remark: we see that $\operatorname{Pow}_{\omega_1}(P) > 0$ if and only if P lies strictly between A and B. Similarly, $\operatorname{Pow}_{\omega_2}(P) > 0$ if and only if P lies strictly between C and D. Because $\operatorname{Pow}_{\omega_1}(P) = \operatorname{Pow}_{\omega_2}(P)$, we have the good case of Theorem 2.4. Hence, because $PA \cdot PB = PC \cdot PD$, we conclude that A, B, C, D are concyclic. Because lines AB and CD are not parallel, it must also be the case that the points O_1, O_2, O_3 are not collinear. □

We have been very careful in our examples above to check that the power of a point holds in the right direction, and to treat the two cases "concurrent" or "all parallel". In practice, this is more rarely an issue, because the specific configuration in an olympiad problem often excludes such pathological configurations. Perhaps one notable exception is USAMO 2009/1 (Example 2.21).

To conclude this section, here is one interesting application of the radical axis that is too surprising to be excluded.

Proposition 2.10. *In a triangle ABC, the circumcenter exists. That is, there is a point O such that $OA = OB = OC$.*

Proof. Construct a circle of radius zero (!) centered at A, and denote it by ω_A. Define ω_B and ω_C similarly. Because the centers are not collinear, we can find their radical center O.

Now we know the powers from O to each of ω_A, ω_B, ω_C are equal. Rephrased, the (squared) length of the "tangents" to each circle are equal: that is, $OA^2 = OB^2 = OC^2$. (To see that OA^2 really is the power, just use $\text{Pow}_{\omega_A}(O) = OA^2 - 0^2 = OA^2$.) From here we derive that $OA = OB = OC$, as required. □

Of course, the radical axes are actually just the perpendicular bisectors of the sides. But this presentation was simply too surprising to forgo. This may be the first time you have seen a circle of radius zero; it will not be the last.

Problems for this Section

Lemma 2.11. *Let ABC be a triangle and consider a point P in its interior. Suppose that $\overline{BC}$ is tangent to the circumcircles of triangles ABP and ACP. Prove that ray AP bisects $\overline{BC}$.*

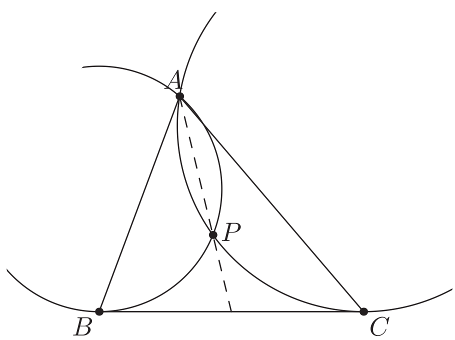

Figure 2.3D. Diagram for Lemma 2.11.

Problem 2.12. Show that the orthocenter of a triangle exists using radical axes. That is, if $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of a triangle ABC, show that the altitudes are concurrent.
Hint: 367

2.4 Coaxial Circles

If a set of circles have the same radical axes, then we say they are **coaxial**. A collection of such circles is called a **pencil** of coaxial circles. In particular, if circles are coaxal, their centers are collinear. (The converse is not true.)

Coaxial circles can arise naturally in the following way.

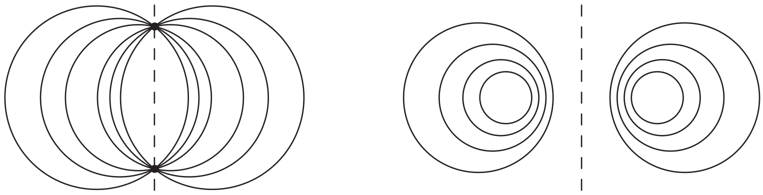

Figure 2.4A. Two pencils of coaxial circles.

Lemma 2.13 (Finding Coaxial Circles). *Three distinct circles Ω_1, Ω_2, Ω_3 pass through a point X. Then their centers are collinear if and only if they share a second common point.*

Proof. Both conditions are equivalent to being coaxial. □

2.5 Revisiting Tangents: The Incenter

We consider again an angle bisector. See Figure 2.5A.

For any point P on the angle bisector, the distances from P to the sides are equal. Consequently, we can draw a circle centered at P tangent to the two sides. Conversely, the two tangents to any circle always have equal length, and the center of that circle lies on the corresponding angle bisector.

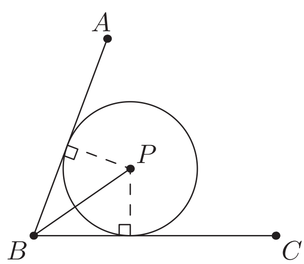

Figure 2.5A. Two tangents to a circle.

From these remarks we can better understand the incenter.

Proposition 2.14. *In any triangle ABC, the angle bisectors concur at a point I, which is the center of a circle inscribed in the triangle.*

Proof. Essentially we are going to complete Figure 2.5A to obtain Figure 2.5B. Let the angle bisectors of $\angle B$ and $\angle C$ intersect at a point I. We claim that I is the desired incenter.

Let D, E, F be the projections of I onto $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Because I is on the angle bisector of $\angle B$, we know that $IF = ID$. Because I is on the angle bisector of $\angle C$, we know that $ID = IE$. (If this reminds you of the proof of the radical center, it should!) Therefore, $IE = IF$, and we deduce that I is also on the angle bisector of $\angle A$. Finally, the circle centered at I with radius $ID = IE = IF$ is evidently tangent to all sides. □

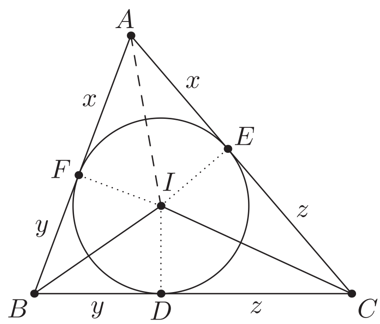

Figure 2.5B. Describing the incircle of a triangle.

The triangle DEF is called the **contact triangle** of $\triangle ABC$.

We can say even more. In Figure 2.5B we have marked the equal lengths induced by the tangents as x, y, and z. Considering each of the sides, this gives us a system of equations of three variables

$$y + z = a$$

$$z + x = b$$

$$x + y = c.$$

Now we can solve for x, y, and z in terms of a, b, c. This is left as an exercise, but we state the result here. (Here $s = \frac{1}{2}(a + b + c)$.)

Lemma 2.15 (Tangents to the Incircle). *If DEF is the contact triangle of $\triangle ABC$, then $AE = AF = s - a$. Similarly, $BF = BD = s - b$ and $CD = CE = s - c$.*

Problem for this Section

Problem 2.16. Prove Lemma 2.15.

2.6 The Excircles

In Lemma 1.18 we briefly alluded the excenter of a triangle. Let us consider it more completely here. The ***A*-excircle** of a triangle ABC is the circle that is tangent to $\overline{BC}$, the extension of $\overline{AB}$ past B, and the extension of $\overline{AC}$ past C. See Figure 2.6A. The ***A*-excenter**, usually denoted I_A, is the center of the A-excircle. The B-excircle and C-excircles are defined similarly and their centers are unsurprisingly called the B-excenter and the C-excenter.

We have to actually check that the A-excircle exists, as it is not entirely obvious from the definition. The proof is exactly analogous to that for the incenter, except with the angle bisector from B replaced with an **external angle bisector**, and similarly for C. As a simple corollary, the incenter of ABC lies on $\overline{AI_A}$.

Now let us see if we can find similar length relations as in the incircle. Let X be the tangency point of the A-excircle on $\overline{BC}$ and B_1 and C_1 the tangency points to rays AB and

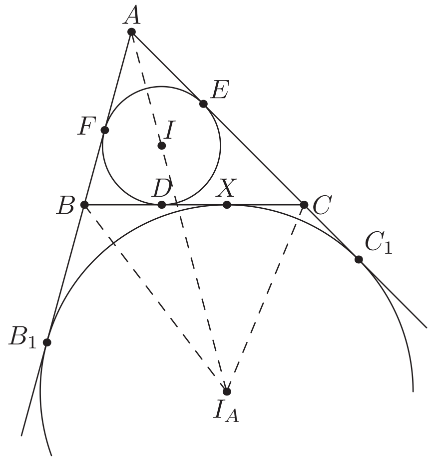

Figure 2.6A. The incircle and A-excircle.

AC. We know that $AB_1 = AC_1$ and that

$$\begin{aligned} AB_1 + AC_1 &= (AB + BB_1) + (AC + CC_1) \\ &= (AB + BX) + (AC + CX) \\ &= AB + AC + BC \\ &= 2s. \end{aligned}$$

We have now obtained the following.

Lemma 2.17 (Tangents to the Excircle). *If AB_1 and AC_1 are the tangents to the A-excircle, then $AB_1 = AC_1 = s$.*

Let us make one last remark: in Figure 2.6A, the triangles AIF and AI_AB_1 are directly similar. (Why?) This lets us relate the ***A*-exradius**, or the radius of the excircle, to the other lengths in the triangle. This exradius is usually denoted r_a. See Lemma 2.19.

Problems for this Section

Problem 2.18. Let the external angle bisectors of B and C in a triangle ABC intersect at I_A. Show that I_A is the center of a circle tangent to $\overline{BC}$, the extension of $\overline{AB}$ through B, and the extension of $\overline{AC}$ through C. Furthermore, show that I_A lies on ray AI.

Lemma 2.19 (Length of Exradius). *Prove that the A-exradius has length*

$$r_a = \frac{s}{s-a} r.$$

Hint: 302

Lemma 2.20. *Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to $\overline{BC}$ at X and D, respectively. Show that $BX = CD$ and $BD = CX$.*

2.7 Example Problems

We finish this chapter with several problems, which we feel are either instructive, classical, or too surprising to not be shared.

Example 2.21 (USAMO 2009/1). Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

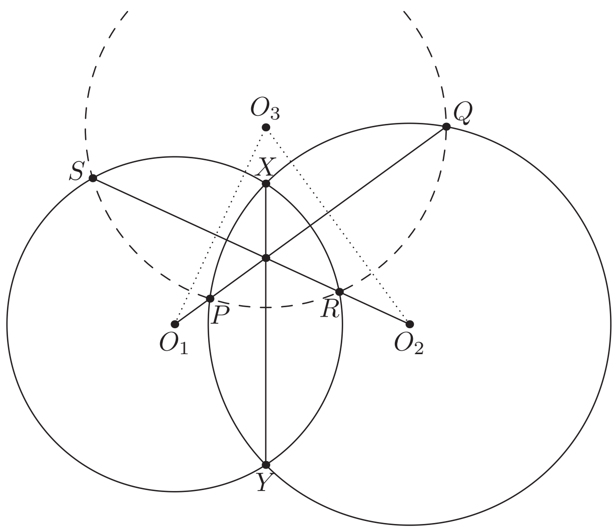

Figure 2.7A. The first problem of the 2009 USAMO.

This was actually a very nasty USAMO problem, in the sense that it was easy to lose partial credit. We will see why.

Let O_3 and ω_3 be the circumcenter and circumcircle, respectively, of the cyclic quadrilateral $PQRS$. After drawing the diagram, we are immediately reminded of our radical axes. In fact, we already know that that lines PQ, RS, and XY concur at a point X, by Theorem 2.9. Call this point H.

Now, what else do we know? Well, glancing at the diagram* it appears that $\overline{O_1O_3} \perp \overline{RS}$. And of course this we know is true, because $\overline{RS}$ is the radical axis of ω_1 an ω_3. Similarly, we notice that $\overline{PQ}$ is perpendicular to O_1O_3.

Focus on $\triangle O_1O_2O_3$. We see that H is its orthocenter. Therefore the altitude from O_3 to $\overline{O_1O_2}$ must pass through H. But line XY is precisely that altitude: it passes through H and is perpendicular to $\overline{O_1O_2}$. Hence, O_3 lies on line XY, and we are done.

Or are we?

Look at Theorem 2.9 again. In order to apply it, we need to know that O_1, O_2, O_3 are not collinear. Unfortunately, this is not always true—see Figure 2.7B.

Fortunately, noticing this case is much harder than actually doing it. We use phantom points. Let O be the midpoint of $\overline{XY}$. (We pick this point because we know this is where O_3

* And you are drawing large scaled diagrams, right?

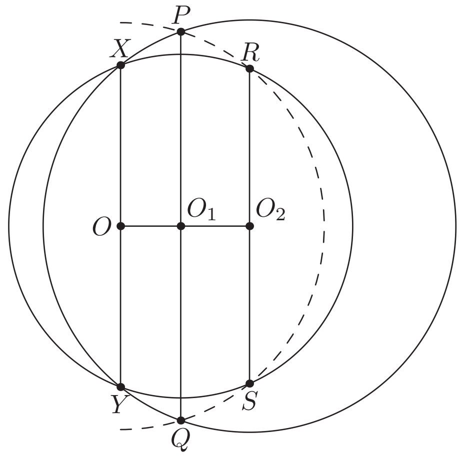

Figure 2.7B. An unnoticed special case.

must be for the problem to hold.) Now we just need to show that $OP = OQ = OR = OS$, from which it will follow that $O = O_3$.

This looks much easier. It should seem like we should be able to compute everything using just repeated applications of the Pythagorean theorem (and the definition of a circle). Trying this,

$$\begin{aligned} OP^2 &= OO_1^2 + O_1P^2 \\ &= OO_1^2 + (O_2P^2 - O_1O_2^2) \\ &= OO_1^2 + r_2^2 - O_1O_2^2. \end{aligned}$$

Now the point P is gone from the expression, but the r_2 needs to go if we hope to get a symmetric expression. We can get rid of it by using $O_2X = r_2 = \sqrt{XO^2 + OO_2^2}$.

$$\begin{aligned} OP^2 &= OO_1^2 + (O_2X^2 + OX^2) - O_1O_2^2 \\ &= OX^2 + OO_1^2 + OO_2^2 - O_1O_2^2 \\ &= \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2. \end{aligned}$$

This is symmetric; the exact same calculations with Q, R, and S yield the same results. We conclude $OP^2 = OQ^2 = OR^2 = OS^2 = \left(\frac{1}{2}XY\right)^2 + OO_1^2 + OO_2^2 - O_1O_2^2$ as desired.

Having presented the perhaps more natural solution above, here is a solution with a more analytic flavor. It carefully avoids the configuration issues in the first solution.

Solution to Example 2.21. Let r_1, r_2, r_3 denote the circumradii of ω_1, ω_2, and ω_3, respectively.

We wish to show that O_3 lies on the radical axis of ω_1 and ω_2. Let us encode the conditions using power of a point. Because O_1 is on the radical axis of ω_2 and ω_3,

$$\begin{aligned} &\text{Pow}_{\omega_2}(O_1) = \text{Pow}_{\omega_3}(O_1) \\ \Rightarrow\ &O_1O_2^2 - r_2^2 = O_1O_3^2 - r_3^2. \end{aligned}$$

Similarly, because O_2 is on the radical axis of ω_1 and ω_3, we have

$$\begin{aligned} &\text{Pow}_{\omega_1}(O_2) = \text{Pow}_{\omega_3}(O_2) \\ \Rightarrow\ &O_1O_2^2 - r_1^2 = O_2O_3^2 - r_3^2. \end{aligned}$$

Subtracting the two gives

$$\begin{aligned} (O_1O_2^2 - r_2^2) - (O_1O_2^2 - r_1^2) &= (O_1O_3^2 - r_3^2) - (O_2O_3^2 - r_3^2) \\ \Rightarrow r_1^2 - r_2^2 &= O_1O_3^2 - O_2O_3^2 \\ \Rightarrow O_2O_3^2 - r_2^2 &= O_1O_3^2 - r_1^2 \\ \Rightarrow \text{Pow}_{\omega_2}(O_3) &= \text{Pow}_{\omega_1}(O_3) \end{aligned}$$

as desired. □

The main idea of this solution is to encode everything in terms of lengths using the radical axis. Effectively, we write down the givens as equations. We also write the desired conclusion as an equation, namely $\text{Pow}_{\omega_2}(O_3) = \text{Pow}_{\omega_1}(O_3)$, then forget about geometry and do algebra. It is an unfortunate irony of olympiad geometry that analytic solutions are often immune to configuration issues that would otherwise plague traditional solutions.

The next example is a classical result of Euler.

Lemma 2.22 (Euler's Theorem). *Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R-2r)$. In particular, $R \geq 2r$.*

The first thing we notice is that the relation is equivalent to proving $R^2 - OI^2 = 2Rr$. This is power of a point, clear as day. So, we let ray AI hit the circumcircle again at L. Evidently we just need to show

$$AI \cdot IL = 2Rr.$$

This looks much nicer to work with—noticing the power expressions gave us a way to clean up the problem statement, and gives us some structure to work on.

We work backwards for a little bit. The final condition appears like similar triangles. So perhaps we may rewrite it as

$$\frac{AI}{r} = \frac{2R}{IL}.$$

There are not too many ways the left-hand side can show up like that. We drop the altitude from I to $\overline{AB}$ as F. Then $\triangle AIF$ has the ratios that we want. (You can also drop the foot to

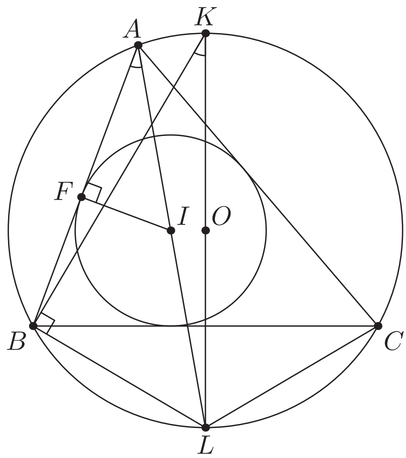

Figure 2.7C. Proving Euler's theorem.

$\overline{AC}$, but this is the same thing.) All that remains is to construct a similar triangle with the lengths $2R$ and IL. Unfortunately, $\overline{IL}$ does not play well in this diagram.

But we hope that by now you recognize $\overline{IL}$ from Lemma 1.18! Write $BL = IL$. Then let K be the point such that $\overline{KL}$ is a diameter of the circle. Then $\triangle KBL$ has the dimensions we want. Could the triangles in question be similar? Yes: $\angle KBL$ and $\angle AFI$ are both right angles, and $\angle BAL = \angle BKL$ by cyclic quadrilaterals. Hence this produces $AI \cdot IL = 2Rr$ and we are done.

As usual, this is not how a solution should be written up in a contest. Solutions should be only written forwards, and without explaining where the steps come from.

Solution to Lemma 2.22. Let ray AI meet the circumcircle again at L and let K be the point diametrically opposite L. Let F be the foot from I to $\overline{AB}$. Notice that $\angle FAI = \angle BAL = \angle BKL$ and $\angle AFI = \angle KBL = 90°$, so

$$\frac{AI}{r} = \frac{AI}{IF} = \frac{KL}{LB} = \frac{2R}{LI}$$

and hence $AI \cdot IL = 2Rr$. Because I lies inside $\triangle ABC$, we deduce the power of I with respect to (ABC) is $2Rr = R^2 - OI^2$. Consequently, $OI^2 = R(R - 2r)$. □

The construction of the diameter appears again in Chapter 3, when we derive the extended law of sines, Theorem 3.1.

Our last example is from the All-Russian Mathematical Olympiad, whose solution is totally unexpected. Please ponder it before reading the solution.

Example 2.23 (Russian Olympiad 2010). Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

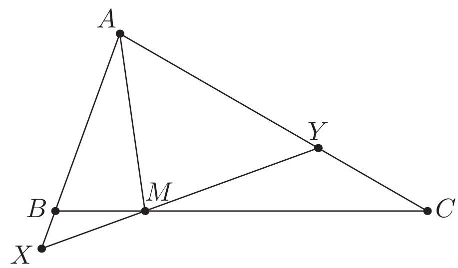

Figure 2.7D. A problem from the All-Russian MO 2010.

What strange conditions have been given. We are told the lengths $AX = AY = 1$ and the perimeter of $\triangle ABC$ is 4, and effectively nothing else. The conclusion, which is an either-or statement, is equally puzzling.

Let us reflect the point A over both X and Y to two points U and V so that $AU = AV = 2$. This seems slightly better, because $AU = AV = 2$ now, and the "two" in the perimeter is now present. But what do we do? Recalling that $s = 2$ in the triangle, we find that U and V are the tangency points of the excircle, call it Γ_a. Set I_A the excenter, tangent to $\overline{BC}$ at T. See Figure 2.7E.

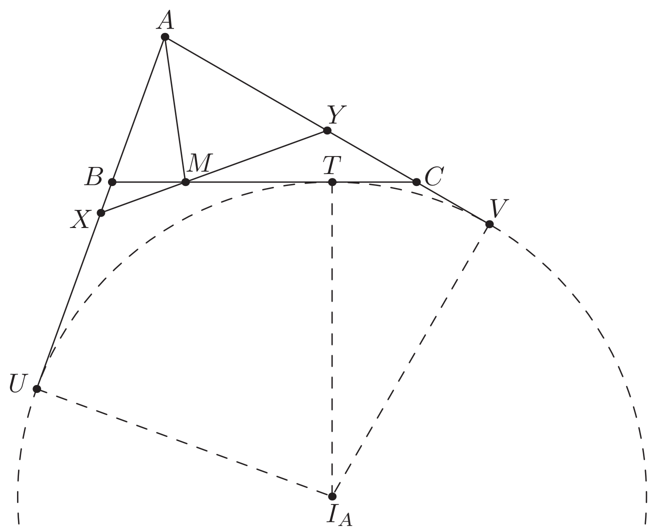

Figure 2.7E. Adding an excircle to handle the conditions.

Looking back, we have now encoded the $AX = AY = 1$ condition as follows: X and Y are the midpoints of the tangents to the A-excircle. We need to show that one of $\triangle ABM$ or $\triangle ACM$ has perimeter equal to the length of the tangent.

Now the question is: how do we use this?

Let us look carefully again at the diagram. It would seem to suggest that in this case, $\triangle ABM$ is the one with perimeter two (and not $\triangle ACM$). What would have to be true in order to obtain the relation $AB + BM + MA = AU$? Trying to bring the lengths closer to the triangle in question, we write $AU = AB + BU = AB + BT$. So we would need $BM + MA = BT$, or $MA = MT$.

So it would appear that the points X, M, Y have the property that their distance to A equals the length of their tangents to the A-excircle. This motivates a last addition to our diagram: construct a circle of radius zero at A, say ω_0. Then X and Y lie on the radical axis of ω_0 and Γ_a; hence so does M! Now we have $MA = MT$, as required.

Now how does the either-or condition come in? Now it is clear: it reflects whether T lies on $\overline{BM}$ or $\overline{CM}$. (It must lie in at least one, because we are told that M lies inside the segment $\overline{BC}$, and the tangency points of the A-excircle to $\overline{BC}$ always lie in this segment as well.) This completes the solution, which we present concisely below.

Solution to Example 2.23. Let I_A be the center of the A-excircle, tangent to $\overline{BC}$ at T, and to the extensions of $\overline{AB}$ and $\overline{AC}$ at U and V. We see that $AU = AV = s = 2$. Then $\overline{XY}$ is the radical axis of the A-excircle and the circle of radius zero at A. Therefore $AM = MT$.

Assume without loss of generality that T lies on $\overline{MC}$, as opposed to $\overline{MB}$. Then $AB + BM + MA = AB + BM + MT = AB + BT = AB + BU = AU = 2$ as desired. □

While we have tried our best to present the solution in a natural way, it is no secret that this is a hard problem by any standard. It is fortunate that such pernicious problems are rare.

2.8 Problems

Lemma 2.24. *Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.* **Hints**: 564 103

Theorem 2.25 (The Pitot Theorem). *Let $ABCD$ be a quadrilateral. If a circle can be inscribed*[†] *in it, prove that $AB + CD = BC + DA$.* **Hint**: 467

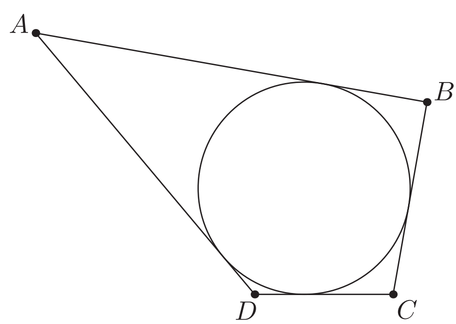

Figure 2.8A. The Pitot theorem: $AB + CD = BC + DA$.

[†] The converse of the Pitot theorem is in fact also true: if $AB + CD = BC + DA$, then a circle can be inscribed inside $ABCD$. Thus, if you ever need to prove $AB + CD = BC + DA$, you may safely replace this with the "inscribed" condition.

Problem 2.26 (USAMO 1990/5). An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle. **Hints**: 260 73 409 **Sol:** p.244

Problem 2.27 (BAMO 2012/4). Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N. **Hints**: 57 77
(b) Prove that no matter where one chooses the point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane. **Hints**: 230 654

Problem 2.28 (JMO 2012/1). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic. **Hints**: 435 601 537 122

Problem 2.29 (IMO 2008/1). Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic. **Hints**: 82 597 **Sol:** p.244

Problem 2.30 (USAMO 1997/2). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively are concurrent. **Hints**: 596 2 611

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. **Hints**: 49 159 134

Problem 2.32 (USAMO 1998/2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC. **Hints**: 659 355 482

Problem 2.33 (IMO 2000/1). Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$. **Hints**: 17 174

Problem 2.34 (Canada 1990/3). Let $ABCD$ be a cyclic quadrilateral whose diagonals meet at P. Let W, X, Y, Z be the feet of P onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively. Show that $WX + YZ = XY + WZ$. **Hints**: 1 414 440 **Sol:** p.245

Problem 2.35 (IMO 2009/2). Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$. **Hints**: 78 544 346

Problem 2.36. Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that (AOD), (BOE), and (COF) intersect at point X other than O. **Hints**: 553 79 **Sol:** p.245

Problem 2.37 (Canada 2007/5). Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let ω, ω_1, ω_2, and ω_3 denote the circumcircles of triangles ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R.

(a) Prove that ω_1, ω_2, and ω_3 intersect in a common point.
(b) Show that lines PD, QE, and RF are concurrent. **Hints**: 376 548 660

Problem 2.38 (Iran TST 2011/1). In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $\overline{TA} \parallel \overline{BC}$. Prove that $TA = TM$. **Hints**: 297 495 154 **Sol:** p.246