CHAPTER **8**

Inversion

Out of nothing I have created a strange new universe. János Bolyai

In this chapter we discuss the method of inversion in the plane. This technique is useful for turning circles into lines and for handling tangent figures.

8.1 Circles are Lines

A **cline** (or **generalized circle**) refers to either a circle or a line. Throughout the chapter, we use "circle" and "line" to refer to the ordinary shapes, and "cline" when we wish to refer to both.

The idea is to view every line as a circle with infinite radius. We add a special point P_∞ to the plane, which every ordinary line passes through (and no circle passes through). This is called the **point at infinity**. Therefore, every choice of three distinct points determines a unique cline—three ordinary points determine a circle, while two ordinary points plus the point at infinity determine a line.

With this said, we can now define an inversion. Let ω be a circle with center O and radius R. We say an **inversion** about ω is a map (that is, a transformation) which does the following.

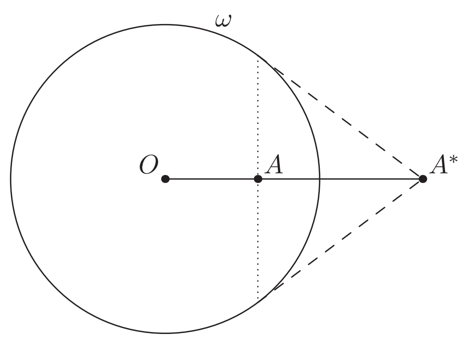

Figure 8.1A. A^* is the image of the point A when we take an inversion about ω.

- The center O of the circle is sent to P_∞.
- The point P_∞ is sent to O.

- For any other point A, we send A to the point A^* lying on ray OA such that $OA \cdot OA^* = r^2$.

Try to apply the third rule to $A = O$ and $A = P_\infty$, and the motivation for the first two rules becomes much clearer. The way to remember it is "$\frac{r^2}{0} = \infty$" and "$\frac{r^2}{\infty} = 0$".

At first, this rule seems arbitrary and contrived. What good could it do? First, we make a few simple observations.

1. A point A lies on ω if and only if $A = A^*$. In other words, the points of ω are fixed.
2. Inversion swaps pairs of points. In other words, the inverse of A^* is A itself. In still other words, $(A^*)^* = A$.

We can also find a geometric interpretation for this mapping, which provides an important setting in which inverses arise naturally.

Lemma 8.1 (Inversion and Tangents). *Let A be a point inside ω, other than O, and A^* be its inverse. Then the tangents from A^* to ω are collinear with A.*

This configuration is shown in Figure 8.1A. It is a simple exercise in similar triangles: just check that $OA \cdot OA^* = r^2$.

This is all fine and well, but it does not provide any clue why we should care about inversion. Inversion is not very interesting if we only look at one point at a time—how about two points A and B?

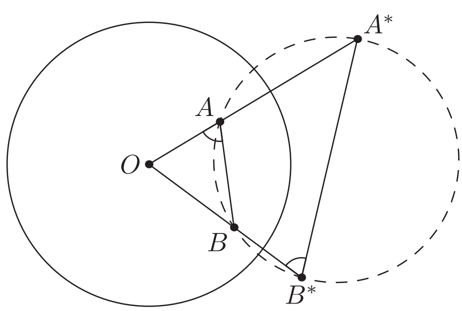

Figure 8.1B. Inversion preserves angles, kind of.

This situation is shown in Figure 8.1B. Now we have some more structure. Because $OA \cdot OA^* = OB \cdot OB^* = r^2$, by power of a point we see that quadrilateral ABB^*A^* is cyclic. Hence we obtain the following theorem.

Theorem 8.2 (Inversion and Angles). *If A^* and B^* are the inverses of A and B under inversion centered at O, then $\measuredangle OAB = -\measuredangle OB^*A^*$.*

Unfortunately, this does not generalize nicely* to arbitrary angles, as the theorem only handles angles with one vertex at O.

It is worth remarking how unimportant the particular value of r has been so far. Indeed, we see that often the radius is ignored altogether; in this case, we refer to this as **inversion**

* The correct generalization is to define an angle between two clines to be the angle formed by the tangents at an intersection point. This happens to be preserved under inversion. However, this is in general not as useful.

around P, meaning that we invert with respect to a circle centered at P with any positive radius. (After all, scaling r is equivalent to just applying a homothety with ratio r^2.)

Problem for this Section

Problem 8.3. If z is a nonzero complex number, show that the inverse of z with respect to the unit circle is $(\overline{z})^{-1}$.

8.2 Where Do Clines Go?

So far we have derived only a few very basic properties of inversion, nothing that would suggest it could be a viable method of attack for a problem. The results of this section will change that.

Rather than looking at just one or two points, we consider entire clines. The simplest example is a just a line through O.

Proposition 8.4. *A line passing through O inverts to itself.*

By this we mean that if we take each point on a line ℓ (including O and P_∞) and invert it, then look at the resulting locus of points, we get ℓ back again. The proof is clear.

What about a line not passing through O? Surprisingly, it is a circle! See Figure 8.2A

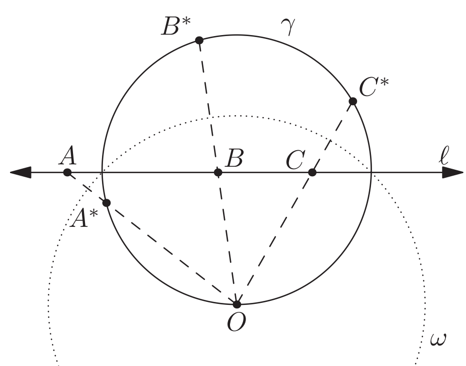

Figure 8.2A. A line inverts to a circle through O, and vice versa.

Proposition 8.5. *The inverse of a line ℓ not passing through O is a circle γ passing through O. Furthermore, the line through O perpendicular to ℓ passes through the center of γ.*

Proof. Let ℓ^* be the inverse of our line. Because P_∞ lies on ℓ, we must have O on ℓ^*. We show ℓ^* is a circle.

Let A, B, C be any three points on ℓ. It suffices to show that O, A^*, B^*, C^* are concyclic. This is easy enough. Because they are collinear, $\measuredangle OAB = \measuredangle OAC$. Using Theorem 8.2, $\measuredangle OB^*A^* = \measuredangle OC^*A^*$, as desired. Since any four points on ℓ^* are concyclic, that implies ℓ^* is just a circle.

It remains to show that ℓ is perpendicular to the line passing through the centers of ω (the circle we are inverting about) and γ. This is not hard to see in the picture. For a proof, let X be the point on ℓ closest to O (so $\overline{OX} \perp \ell$). Then X^* is the point on γ farthest from O, so that $\overline{OX^*}$ is a diameter of γ. Since O, X, X^* are collinear by definition, this implies the result. □

In a completely analogous fashion one can derive the converse—the image of a circle passing through O is a line. Also, notice how the points on ω are fixed during the whole transformation.

This begs the question—what happens to the other circles? It turns out that these circles also invert to circles. Our proof here is of a different style than the previous one (although the previous proof can be rewritten to look more like this one). Refer to Figure 8.2B.

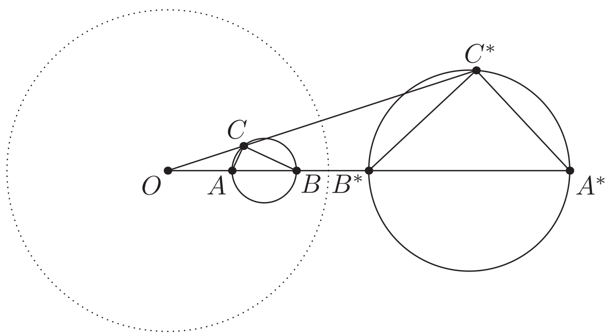

Figure 8.2B. A circle inverts to another circle.

Proposition 8.6. *Let γ be a circle not passing through O. Then γ^* is also a circle and does not contain O.*

Proof. Because neither O nor P_∞ is on γ, the inverse γ^* cannot contain these points either. Now, let $\overline{AB}$ be a diameter of γ with O on line AB (and $A, B \neq O$). It suffices to prove that γ^* is a circle with diameter $\overline{A^*B^*}$.

Consider any point C on γ. Observe that

$$90^\circ = \measuredangle BCA = -\measuredangle OCB + \measuredangle OCA.$$

By Theorem 8.2, we see that $-\measuredangle OCA = \measuredangle OA^*C^*$ and $-\measuredangle OCB = \measuredangle OB^*C^*$. Hence, a quick angle chase gives

$$90^\circ = \measuredangle OB^*C^* - \measuredangle OA^*C^* = \measuredangle A^*B^*C^* - \measuredangle B^*A^*C^* = -\measuredangle B^*C^*A^*$$

and hence C^* lies on the circle with diameter $\overline{A^*B^*}$. By similar work, any point on γ^* has inverse lying on γ, and we are done. □

It is worth noting that the centers of these circles are also collinear. (However, keep in mind that the centers of the circle do not map to each other!)

We can summarize our findings in the following lemma.

Theorem 8.7 (Images of Clines). *A cline inverts to a cline. Specifically, in an inversion through a circle with center O,*

(a) A line through O inverts to itself.
(b) A circle through O inverts to a line (not through O), and vice versa. The diameter of this circle containing O is perpendicular to the line.
(c) A circle not through O inverts to another circle not through O. The centers of these circles are collinear with O.

We promised that inversion gives the power to turn circles into lines. This is a result of (b)—if we invert through a point with many circles, then all those circles become lines.

Finally, one important remark. Tangent clines (that is, clines which intersect exactly once, including at P_∞ in the case of two lines) remain tangent under inversion. This has the power to send tangent circles to parallel lines—we simply invert around the point at which they are internally or externally tangent.

Problems for this Section

Problem 8.8. In Figure 8.2C, sketch the inverse of the five solid clines (two lines and three circles) about the dotted circle ω. **Hint**: 279

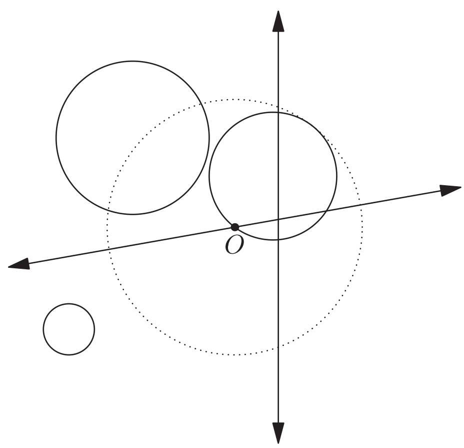

Figure 8.2C. Practice inverting.

Lemma 8.9 (Inverting an Orthocenter). *Let ABC be a triangle with orthocenter H and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Perform an inversion around C with radius $\sqrt{CH \cdot CF}$. Where do the six points each go?* **Hint**: 257

Lemma 8.10 (Inverting a Circumcenter). *Let ABC be a triangle with circumcenter O. Invert around C with radius 1. What is the relation between O^*, C, A^*, and B^*?* **Hint**: 252

Lemma 8.11 (Inverting the Incircle). *Let ABC be a triangle with circumcircle Γ and contact triangle DEF. Consider an inversion with respect to the incircle of triangle ABC. Show that Γ is sent to the nine-point circle of triangle DEF.* **Hint**: 560

8.3 An Example from the USAMO

An example at this point would likely be illuminating. We revisit a problem first given in Chapter 3.

Example 8.12 (USAMO 1993/2). Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic.

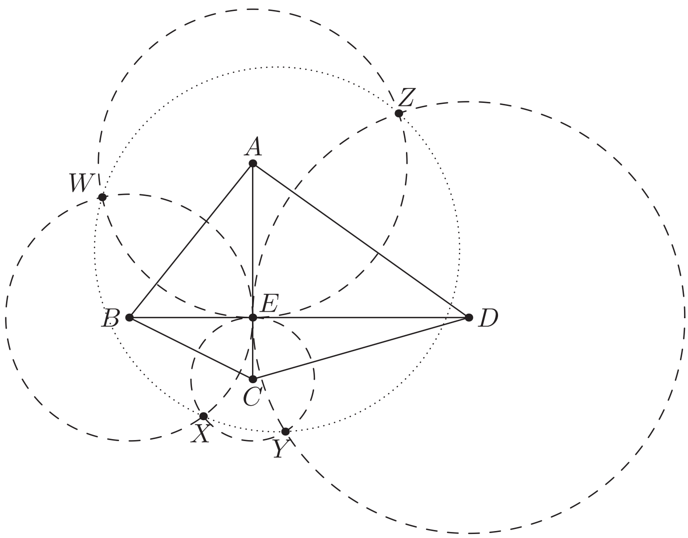

Figure 8.3A. Adding in some circles.

Let the reflections respectively be W, X, Y, Z.

At first, this problem seems a strange candidate for inversion. Indeed, there are no circles. Nevertheless, upon thinking about the reflection condition one might notice

$$AW = AE = AZ$$

which motivates us to construct a circle ω_A centered at A passing through all three points. If we define ω_B, ω_C, and ω_D similarly, suddenly we no longer have to worry about reflections. W is the just the second intersection of ω_A and ω_B, and so on.

Let us rephrase this problem in steps now.

1. Let $ABCD$ be a quadrilateral with perpendicular diagonals that meet at E.
2. Let ω_A be a circle centered at A through E.
3. Define ω_B, ω_C, ω_D similarly.
4. Let W be the intersection of ω_A and ω_B other than E.
5. Define X, Y, Z similarly.
6. Prove that $WXYZ$ is concyclic.

At this point, it may not be clear why we want to invert. Many students learning inversion for the first time are tempted to invert about ω_A. As far as I can tell, this leads nowhere, because it misses out on one of the most compelling reasons to invert:

Inversion lets us turn circles into lines.

This is why inversion around ω_A seems fruitless. There are few (read: zero) circles passing through A, so all the circles in the figure stay as circles, while some former lines become new circles. Hence inverting about ω_A is counterproductive: the resulting problem is more complicated than the original!

So what point has a lot of circles passing through it? Well, how about E? All four circles pass through it. Hence, we invert around a circle centered at E with radius 1. (Just because a point has no circle around it does not prevent us from using it as the center of inversion!)

What happens to each of the mapped points? Let us consider it step-by-step.

1. $A^*B^*C^*D^*$ is still some quadrilateral. As A^*, and C^* stay on line AC, and B^* and D^* stay on line BD, we have that $A^*B^*C^*D^*$ also has perpendicular diagonals meeting at E. Since $ABCD$ is arbitrary, we likewise treat $A^*B^*C^*D^*$ as arbitrary.[†]
2. ω_A passes through E, so it maps to some line perpendicular to line EA. This is not enough information to determine ω_A^* yet—what is the point of intersection ω_A^* has with line EA? Actually, it is the midpoint of $\overline{A^*E}$. For let M_A be the point diametrically opposite E on ω_A; this is the pre-image of the their intersection. Now A is the midpoint of $\overline{M_A E}$, so M_A^* is the midpoint of $\overline{A^*E}$.
   In other words, ω_A^* is the perpendicular bisector of $\overline{A^*E}$.
3. Define ω_B^*, ω_C^*, ω_D^* similarly.
4. W^* is the intersection of the two lines ω_A^* and ω_B^*, simply because W is the intersection of ω_A and ω_B other than E. (Of course, ω_A^* and ω_B^* also meet at the point at infinity, which is the image of E.)
5. X^*, Y^*, Z^* are also defined similarly.
6. We wish to show $WXYZ$ is cyclic. By Theorem 8.7, this is equivalent to showing $W^*X^*Y^*Z^*$ is cyclic.

This is the thought process for inverting a problem. We consider the steps used to construct the original problem, and one by one find their inversive analogs. While perhaps not easy at first, this requires no ingenuity and is a skill that can be picked up with enough practice, since it is really just a mechanical calculation.

Figure 8.3B shows the completed diagram.

We are just moments from finishing. We wish to show that quadrilateral $W^*X^*Y^*Z^*$ is cyclic. But it is a rectangle, so this is obvious!

Solution to Example 8.12. Define ω_A, ω_B, ω_C, ω_D to be circles centered at A, B, C, D passing through E. Observe that W is the second intersection of ω_A and ω_B, et cetera.

[†] Degrees of freedom, anyone? When you are considering the inverted version of a problem, you want to make sure the number of degrees of freedom does not change. See Section 5.3 for more discussion on degrees of freedom.

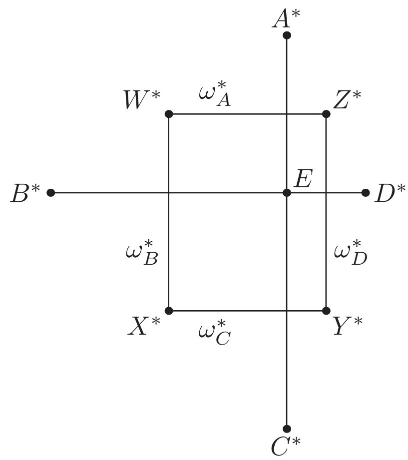

Figure 8.3B. Inverting the USAMO.

Consider an inversion at E. It maps ω_A, ω_B, ω_C, ω_D to four lines which are the sides of a rectangle. Hence the images of W, X, Y, Z under this inversion form a rectangle, which in particular is cyclic. Inverting back, $WXYZ$ is cyclic as desired. □

Notice that we do not have to go through the full detail in explaining how to arrive at the inverted image. In a contest, it is usually permissible to just state the inverted problem, since deriving the inverted figure is a straightforward process.

Usually an inverted problem will not be *this* easy.[‡] However, we often have good reason to believe that the inverted problem is simpler than the original. In the above example, the opportunity to get rid of all the circles motivated our inversion at E, and indeed we found the resulting problem to be trivial.

8.4 Overlays and Orthogonal Circles

Consider two circles ω_1 and ω_2 with centers O_1 and O_2 intersecting at two points X and Y. We say they are **orthogonal** if

$$\angle O_1 X O_2 = 90^\circ,$$

i.e., the lines O_1X and O_1Y are the tangents to the second circle. Of course, ω_1 is orthogonal to ω_2 if and only if ω_2 is orthogonal to ω_1.

It is clear that if ω_2 is a circle and O_1 a point outside it, we can draw a unique circle centered at O_1 orthogonal to ω_2: namely, the circle whose radius is equal to the length of the tangent to ω_2.

Orthogonal circles are nice because of the following lemma.

[‡] But you can certainly find other examples. At the 2014 IMO, one of my teammates said that he was looking for problems that were trivialized by inversion. Another friend responded that this was easy—just take a trivial problem and invert it!

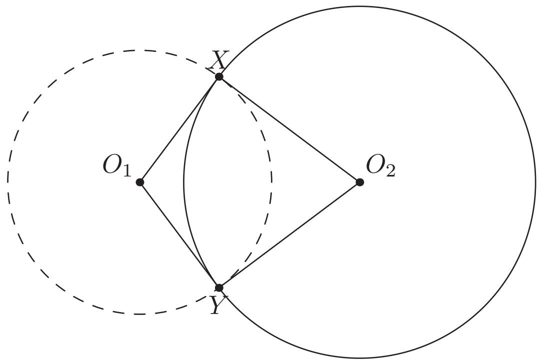

Figure 8.4A. Two orthogonal circles.

Lemma 8.13 (Inverting Orthogonal Circles). *Let ω and γ be orthogonal circles. Then γ inverts to itself under inversion with respect to ω.*

Proof. This is a consequence of power of a point. Let ω and γ intersect at X and Y, and denote by O the center ω. Consider a line through O intersecting γ at A and B. Then

$$OX^2 = OA \cdot OB$$

but since OX is the radius ω, A inverts to B. □

What's the upshot? When a figure inverts to itself, we get to exploit what I call the "inversion overlay principle". Loosely, it goes as follows:

Problems that invert to themselves are usually really easy.

There are a few ways this can happen. Sometimes it is because we force a certain circle to be orthogonal. Other times it is a good choice of radius that plays well with the problem. In either case the point is that we gain information by overlaying the inverted diagram onto the original.

Here is the most classical example of overlaying, called a **Pappus chain** embedded in a **shoemaker's knife**. See Figure 8.4B.

Example 8.14 (Shoemaker's Knife). Let A, B, C be three collinear points (in that order) and construct three semicircles Γ_{AC}, Γ_{AB}, ω_0, on the same side of $\overline{AC}$, with diameters $\overline{AC}$, $\overline{AB}$, $\overline{BC}$, respectively. For each positive integer k, let ω_k be the circle tangent to Γ_{AC} and Γ_{AB} as well as ω_{k-1}.

Let n be a positive integer. Prove that the distance from the center of ω_n to $\overline{AC}$ is n times its diameter.

The point of inverting is to handle the abominable tangency conditions. Note that each ω_i is tangent to both Γ_{AB} and Γ_{AC}, so it makes sense to force both of these circles into lines. This suggests inverting about A. As an added bonus, these two lines become parallel.

It is perhaps not clear yet what to use as the radius, or even if we need to pick a radius. However, we want to ensure that the diameter of ω_n remains a meaningful quantity after the inversion. This suggests keeping ω_n fixed.

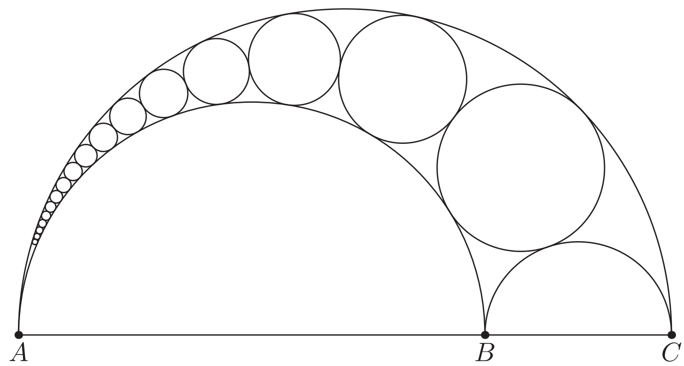

Figure 8.4B. The Shoemaker's Knife.

This motivates us to invert around A with radius r in such a way that ω_n is orthogonal to our circle of inversion. What effect does this have?

- ω_n stays put, by construction.
- The semicircles Γ_{AB} and Γ_{AC} pass through A, so their images Γ^*_{AB} and Γ^*_{AC} are lines perpendicular to line AC.
- All the other ω_i are now circles tangent to these two lines.

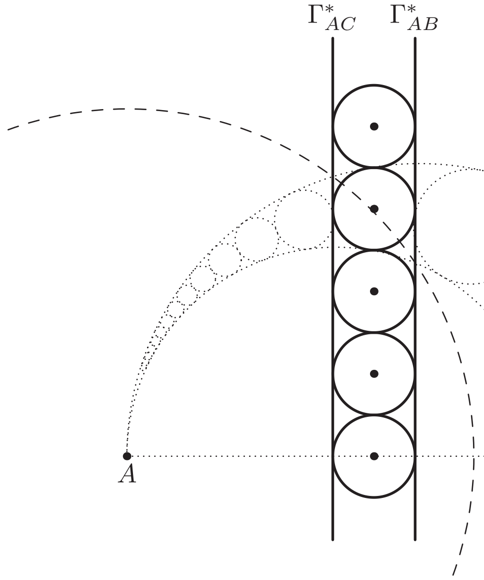

Figure 8.4C. Inverting with ω_3 fixed (so $n = 3$). We invert around the dashed circle centered at A, orthogonal to ω_3.

Figure 8.4C shows the inverted image, overlaid on the original image. The two semicircles have become convenient parallel lines, and the circles of the Pappus chain line up obediently between them. Because the circles are all congruent, the conclusion is now obvious.

8.5 More Overlays

An example of the second type of overlay is the short inversive proof of Lemma 4.33 we promised.

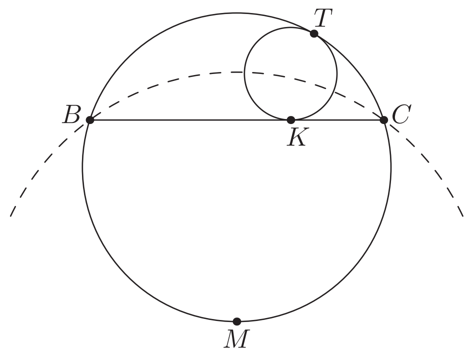

Figure 8.5A. Revisiting Lemma 4.33.

Example 8.15. Let $\overline{BC}$ be a chord of a circle Ω. Let ω be a circle tangent to chord $\overline{BC}$ at T and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc $\widehat{BC}$ not containing T. Moreover, MC^2 is the power of M with respect to ω.

Proof. Let Γ be the circle centered at M passing through B and C. What happens when we invert around Γ?

Firstly, Ω is a circle through M, so it gets sent to a line. Because B and C lie on Γ and are fixed by this inversion, it must be precisely the line BC. In particular, this implies line BC gets sent to Ω. In other words, the inversion simply swaps line BC and Γ.

Perhaps the ending is already obvious. We claim that ω just gets sent to itself. Because $\overline{BC}$ and Ω trade places, ω^* is also a circle tangent to both. Also, the centers of ω^* and ω are collinear with M. This is enough to force $\omega = \omega^*$. (Why?)

Now K is the tangency point of ω with $\overline{BC}$, so K^* is the tangency point of $\omega^* = \omega$ with $(MB^*C^*) = \Omega$. But this is T; hence K and T are inverses.

In particular, M, K, T are collinear and $MK \cdot MT = MC^2$. □

Here is a nice general trick that can force overlays when dealing with a triangle ABC.

Lemma 8.16 (Force-Overlaid Inversion). *Let ABC be a triangle. Consider the transformation consisting of an inversion about A with radius $\sqrt{AB \cdot AC}$, followed by a reflection around the angle bisector of $\angle BAC$. This transformation fixes B and C.*

The above demonstration applies the lemma with $A = M$. Because $\triangle BMC$ was isosceles, there was no need to use the additional reflection.

Fixing a triangle ABC is often very powerful since problems often build themselves around ABC. In particular, tangency to (ABC) is involved (as it becomes tangency to line BC). This led to the solution in the above example.

Problem for this Section

Problem 8.17. Work out the details in the proof of Lemma 8.16.

8.6 The Inversion Distance Formula

The inversion distance formula gives us a way to handle lengths in inversion. It is completely multiplicative, making it nice for use with ratios but more painful if addition is necessary.

Theorem 8.18 (Inversion Distance Formula). *Let A and B be points other than O and consider an inversion about O with radius r. Then*

$$A^*B^* = \frac{r^2}{OA \cdot OB} \cdot AB.$$

Equivalently,

$$AB = \frac{r^2}{OA^* \cdot OB^*} A^*B^*.$$

This first relation follows from the similar triangles we used in Figure 8.1B, and is left as an exercise. The second is a direct consequence of the first (why?).

The inversion distance formula is useful when you need to deal with a bunch of lengths. See Problem 8.20.

Problems for this Section

Problem 8.19. Prove the inversion distance formula.

Problem 8.20 (Ptolemy's Inequality)**.** For any four distinct points A, B, C, and D in a plane, no three collinear, prove that

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Moreover, show that equality holds if and only if A, B, C, D lie on a circle in that order.
Hints: 118 136 539 130

8.7 More Example Problems

The first problem is taken from the Chinese Western Mathematical Olympiad.

Example 8.21 (Chinese Olympiad 2006). Let $ADBE$ be a quadrilateral inscribed in a circle with diameter $\overline{AB}$ whose diagonals meet at C. Let γ be the circumcircle of $\triangle BOD$, where O is the midpoint of $\overline{AB}$. Let F be on γ such that $\overline{OF}$ is a diameter of γ, and let ray FC meet γ again at G. Prove that A, O, G, E are concyclic.

We are motivated to consider inversion by the two circles passing through O, as well as the fact that O itself is a center of a circle through many points. Inversion through O also preserves the diameter $\overline{AB}$, which is of course important.

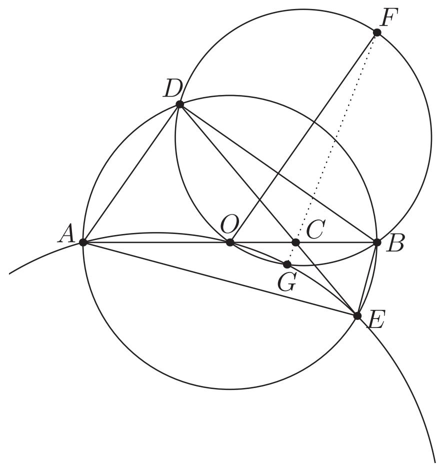

Figure 8.7A. Show that $OAEG$ is concyclic.

Before inverting, though, let us rewrite the problem with phantom point G_1 as the intersection of (OFB) and (OAE), and attempt to prove instead that F, C, G_1 are collinear. This lets us define G_1^* as the intersection of two lines.

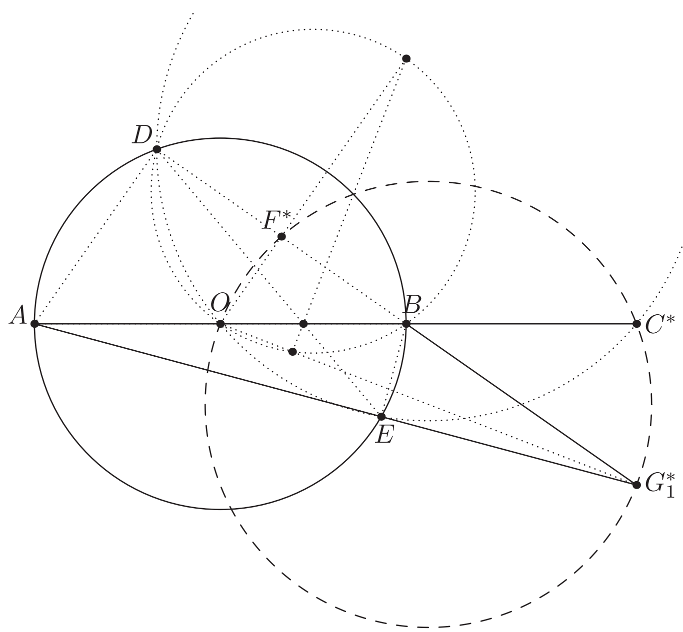

Figure 8.7B. In the inverted image, we wish to show that points O, F^*, C^*, G_1^* are cyclic.

We now invert around the circle with diameter $\overline{AB}$. We figure out where each point goes.

1. Points D, B, A, E stay put, because they lie on the circle we are inverting around. So $D^* = D$, etc.

2. C was the intersection of $\overline{AB}$ and $\overline{DE}$. Hence C^* is a point on line AB so that C^*DOE is cyclic.
3. F is the point diametrically opposite O on (BOD). That means that $\angle ODF = 90^\circ$. So, $\angle OF^*D^* = 90^\circ$. Similarly, $\angle OF^*B^* = 90^\circ$. Hence, F^* is just the midpoint of $\overline{DB}$!
4. G_1 is defined as the intersection of (OFB) and (OAE), so G_1^* is the intersection of lines F^*B and AE.
5. We wish to show that O, F^*, C^*, and G_1 are concyclic.

Okay. Well, $\overline{OF^*} \perp \overline{BD}$; thus to prove O, F^*, C^*, G_1^* are concyclic, it suffices to show that $\overline{G_1^*C^*} \perp \overline{AC^*}$. Now look once more at circle $(OEDC^*)$. Notice something?

Because $\overline{AD} \perp \overline{BG_1^*}$, $\overline{BE} \perp \overline{AG_1^*}$, and O is the midpoint of $\overline{AB}$, we discover this is the nine-point circle of $\triangle ABG_1^*$. We are done.

Solution to Example 8.21. Let G_1 be the intersection of (ODB) and (OAE) and invert around the circle with diameter $\overline{AB}$. In the inverted image, F^* is the midpoint of $\overline{BD}$, C^* lies on line AB and (DOE), and G^* is the intersection of lines DB and AE. We wish to show O, F^*, C^*, G_1^* are cyclic.

Because (OED) is the nine-point circle of $\triangle ABG_1^*$, we see C^* is the foot of G_1^* onto line AB. On the other hand, $\angle OF^*B = 90^\circ$ as well so we are done. □

Let us conclude by examining the fifth problem from the 2009 USA olympiad.

Example 8.22 (USAMO 2009/5). Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersect $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

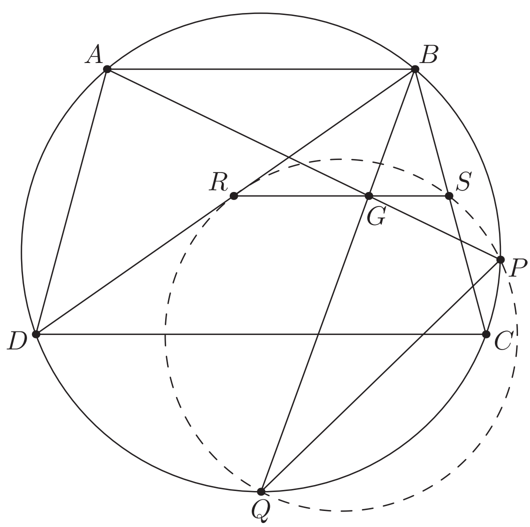

Figure 8.7C. USAMO 2009/5.

The main reason we might want to attempt inversion is that there are not just four, or even five, but six points all lying on one circle. It would be great if we could make that circle into a line.

So if we are going to invert, we should do so around a point on the circle ω. Because we have a bisector at $\angle CBD$, it makes sense to invert around B in order to keep this condition nice. Also, the parallel lines become tangent circles at B. More plainly, there are just a lot of lines passing through B.

Again we work out what happens in steps.

1. Cyclic quadrilateral $ABCD$ becomes a point B and three points A^*, C^*, D^* on a line in that order. Because $\overline{AB} \parallel \overline{CD}$, we actually see that $\overline{A^*B}$ is tangent to (BC^*D^*).
2. G is an arbitrary point inside triangle BCD. That means G^* is some point inside $\angle C^*BD^*$, but outside triangle BC^*D^*.
3. R and S are the intersections of a parallel line through G with $\overline{BD}$ and $\overline{BC}$. Therefore R^* is the intersection of a circle tangent to (BC^*D^*) at B (this is the image of parallel lines) with ray BD^*. S^* is the intersection of this same circle with ray BS^*.
4. Q was the intersection of $(ABCD)$ with ray BG, so now Q^* is the intersection of $\overline{BG^*}$ with the line through A^*, C^*, and D^*.
5. P was the intersection of $(ABCD)$ with line AG. Hence P^* is the point on line A^*C^* such that $BA^*G^*P^*$ is cyclic.
6. We wish to show that $P^*Q^*R^*S^*$ is cyclic if and only if $\overline{BG^*}$ bisects $\angle R^*BS^*$.

The inverted diagram is shown in Figure 8.7D.

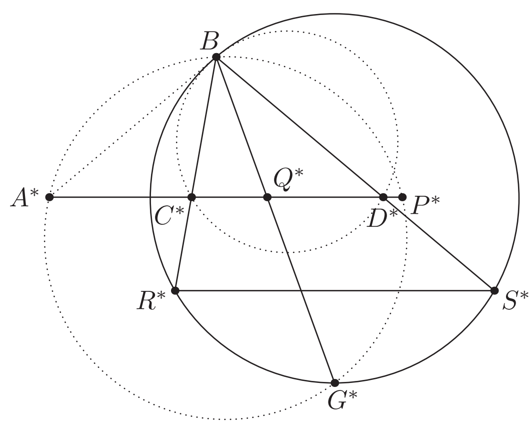

Figure 8.7D. Inverting the USAMO... again!

Now it appears that $\overline{P^*Q^*}$ is parallel to $\overline{S^*R^*}$. Actually, this is obvious, because there is a homothety at B taking $\overline{C^*D^*}$ to $\overline{S^*R^*}$. This is good for us, because now $P^*Q^*R^*S^*$ is cyclic if and only if it is isosceles.

We can also basically ignore (BC^*D^*) now; it is just there to give us these parallel lines. For that matter, we can more or less ignore C^* and D^* now too.

Let us eliminate the point A^*. We have

$$\measuredangle Q^*P^*G^* = \measuredangle A^*P^*G^* = \measuredangle A^*BG^* = \measuredangle BS^*G^*.$$

Seeing this, we extend line G^*P^* to meet (BS^*R^*) at X, as in Figure 8.7E. This way,

$$\measuredangle Q^*P^*G^* = \measuredangle BS^*G^* = \measuredangle BX^*G^*.$$

Therefore, $\overline{P^*Q^*} \parallel \overline{BX}$ holds unconditionally. This lets us get rid of P^* in the sense that it is just a simple intersection of $\overline{G^*X}$ and the parallel line; we can anchor the problem around (BXR^*S^*).

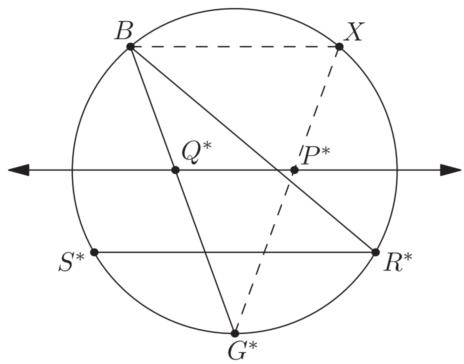

Figure 8.7E. Cleaning up the inverted diagram.

Thus, we have reduced the problem to the following.

> Let BXS^*R^* be an isosceles trapezoid and ℓ a fixed line parallel to its bases. Let G^* be a point on its circumcircle and denote the intersections of ℓ with $\overline{BG^*}$ and $\overline{XG^*}$ by Q^* and P^*. Prove that $P^*S^* = Q^*R^*$ if and only if G^* is the midpoint of arc R^*S^*.

This is actually straightforward symmetry. See the solution below.

Solution to Example 8.22. Perform an inversion around B with arbitrary radius, and denote the inverse of a point Z with Z^*.

After inversion, we obtain a cyclic quadrilateral $BS^*G^*R^*$ and points C^*, D^* on $\overline{BS^*}$, $\overline{BR^*}$, such that (BC^*D^*) is tangent to $(BS^*G^*R^*)$—in other words, so that $\overline{C^*D^*}$ is parallel to $\overline{S^*R^*}$. Point A^* lies on line $\overline{C^*D^*}$ so that $\overline{A^*B}$ is tangent to $(BS^*G^*R^*)$. Points P^* and Q^* are the intersections of (A^*BG^*) and $\overline{BG^*}$ with line C^*D^*.

Observe that $P^*Q^*R^*S^*$ is a trapezoid, so it is cyclic if and only if it isosceles.

Let X be the second intersection of line G^*P^* with (BS^*R^*). Because $\measuredangle Q^*P^*G^* = \measuredangle A^*BG^* = \measuredangle BXG^*$, we find that BXS^*R^* is an isosceles trapezoid.

If G^* is indeed the midpoint of the arc then everything is clear by symmetry now. Conversely, if $P^*R^* = Q^*S^*$ then that means $P^*Q^*R^*S^*$ is a cyclic trapezoid, and hence that the perpendicular bisectors of $\overline{P^*Q^*}$ and $\overline{R^*S^*}$ are the same. Hence B, X, P^*, Q^* are symmetric around this line. This forces G^* to be the midpoint of arc R^*S^* as desired. □

These two examples demonstrate inversion as a means of transforming one problem into another (as opposed to some of the overlaying examples, which used both at once). It is almost like you are given a choice—which of these two problems looks easier, the inverted one or the original one? Which would you like to solve?

8.8 When to Invert

As a reminder, here are things inversion with a center O handles well. Hopefully these were clear from the examples.

- Clines tangent to each other. In particular, we can take a tangent pair of circles to two parallel lines.
- Several circles pass through O. Inverting around O eliminates the circles.
- Diagrams that invert to themselves! Overlaying an inverted diagram is frequently fruitful.

Here are things that inversion does not handle well.

- Scattered angles. Theorem 8.2 gives us control over angles that have a ray passing through a center O, but we do not have much control over general angles.
- Problems that mostly involve lines and not circles.

Finally, here is a reminder of what inversion through a circle ω with center O preserves (and what it does not).

- Points on ω are fixed.
- Clines are sent to clines. Moreover,
    - If a circle γ is mapped to a line ℓ, then ℓ is perpendicular to the line joining O to the center of γ.
    - If a circle γ is mapped to γ^*, the center of γ is not in general the center of γ^*. It is true, however, that the centers of γ and γ^* are collinear with the center of inversion.
- Tangency and intersections are preserved.

8.9 Problems

Problem 8.23. Let ABC be a right triangle with $\angle C = 90^\circ$ and let X and Y be points in the interiors of $\overline{CA}$ and $\overline{CB}$, respectively. Construct four circles passing through C, centered at A, B, X, Y. Prove that the four points lying on at exactly two of these four circles are concyclic. (See Figure 8.9A.) **Hints**: 198 626 178 577

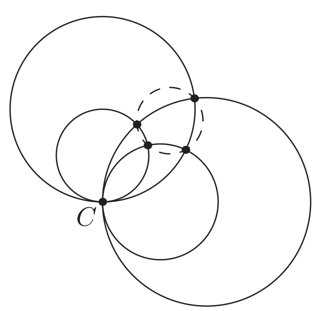

Figure 8.9A. The four intersections are concyclic (dashed circle).

Problem 8.24. Let ω_1, ω_2, ω_3, ω_4 be circles with consecutive pairs tangent at A, B, C, D, as shown in Figure 8.9B. Prove that quadrilateral $ABCD$ is cyclic. **Hints**: 294 677 172 **Sol:** p.272

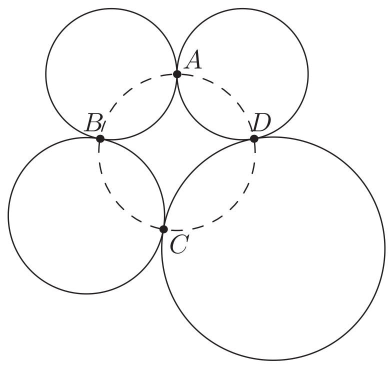

Figure 8.9B. Is there a connection between this and Theorem 2.25?

Problem 8.25. Let A, B, C be three collinear points and P be a point not on this line. Prove that the circumcenters of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ lie on a circle passing through P. **Hints**: 465 536 496

Problem 8.26 (BAMO 2008/6). A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

Hints: 439 170 256

Problem 8.27 (Iran Olympiad 1996). Consider a semicircle with center O and diameter $\overline{AB}$. A line intersects line AB at M and the semicircle at C and D such that $MC > MD$ and $MB < MA$. Suppose (AOC) and (BOD) meet at a point K other than O. Prove that $\angle MKO = 90^\circ$. **Hints**: 403 27 **Sol:** p.272

Problem 8.28 (Shortlist 2003/G4). Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_4, Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Hints: 120 247 22

Problem 8.29. Let ABC be a triangle with incenter I and circumcenter O. Prove that line IO passes through the centroid G_1 of the contact triangle. **Hints**: 532 323 579

Problem 8.30 (NIMO 2014). Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to

$\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$. **Hints**: 362 125 578 663 **Sol:** p.273

Problem 8.31 (EGMO 2013/5). Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q. Prove that $\angle ACP = \angle QCB$. **Hints**: 282 449 255 143 **Sol:** p.273

Problem 8.32 (Russian Olympiad 2009). In triangle ABC with circumcircle Ω, the internal angle bisector of $\angle A$ intersects $\overline{BC}$ at D and Ω again at E. The circle with diameter $\overline{DE}$ meets Ω again at F. Prove that $\overline{AF}$ is a symmedian of triangle ABC. **Hints**: 594 648 321

Problem 8.33 (Shortlist 1997). Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (indices taken mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear. **Hints**: 76 242 620 561

Problem 8.34 (IMO 1993/2). Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio $\frac{AB \cdot CD}{AC \cdot BD}$, and prove that the circumcircles of the triangles ACD and BCD are orthogonal. **Hints**: 7 384 322 3

Problem 8.35 (IMO 1996/2). Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE concur. **Hints**: 581 638 338 341

Problem 8.36 (IMO 2015/3). Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other. **Hints**: 402 673 324 400 155 **Sol:** p.274

Problem 8.37 (ELMO Shortlist 2013). Let ω_1 and ω_2 be two orthogonal circles, and let the center of ω_1 be O. Diameter $\overline{AB}$ of ω_1 is selected so that B lies strictly inside ω_2. The two circles tangent to ω_2 through both O and A touch ω_2 at F and G. Prove that quadrilateral $FOGB$ is cyclic. **Hints**: 96 353 112 **Sol:** p.274