Introduction

Dear God,
If I have just one hour remaining to live,
Please put me in a calculus class
So that it will seem to last forever.
— A bored student's prayer

In the study of mathematics non-routine problems, puzzles, paradoxes, and sophisms often delight and fascinate. Captivating examples can excite, enlighten, and inspire learners and spur their passion for discovery. Furthermore, thought-provoking exercises and contemplation of paradoxes can naturally engage students and offer them a unique opportunity to understand more fully the history and development of mathematics. "Justification of otherwise inexplicable notions on the grounds that they yield useful results has occurred frequently in the evolution of mathematics [15]."

The teaching and learning process often loses its effectiveness for lack of appropriate intellectual challenges and for insufficient active involvement or emotional investment from students in the experience. What we tend to remember most are knowledge and learning experiences tied to intense thinking, noteworthy discovery, or inspired creativity. This book presents problems and examples that may lead students to contemplate conceptual issues in calculus and to comprehend the subtleties of this subject more deeply.

In that light, this book aims to enhance the teaching and learning of a first-year calculus course. The following major topics from a typical single-variable calculus course are explored in the book: functions, limits, derivatives, and integrals.

The book consists of two main types of examples: paradoxes and sophisms. Why should we endeavor to study these atypical or troublesome problems from calculus? Consider these compelling remarks from *How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics* [6, p. 6]: "Logic abhors the ambiguous, the paradoxical, and especially the contradictory, but the creative mathematician welcomes such problematic situations because they raise the question, 'What is going on here?' Thus the problematic signals a situation that is worth investigating. The problematic is a potential source of new mathematics."

The word *paradox* comes from the Greek word *paradoxon* which means *unexpected*. Several usages of this word exist, including those that allow for contradiction. However, in this book, the word paradox will exclusively be used to mean a surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true. "All I know is that I know nothing," a statement attributed to Socrates in Plato's *Republic*, offers a classical paradox of logic. In this book the paradoxes we examine relate to notions in calculus or the study of functions and limits. While some of the paradoxes presented in the book (such as "A cat on a ladder" and "Encircling the Earth") may more naturally be considered precalculus topics, they can still be discussed in calculus classes to demonstrate that intuition can fail, even when we are considering examples with shapes as mundane as circles or spheres. On the other hand, while all the paradoxes or problems we examine can be understood in a first-year calculus class, the explanations for a few problems may involve topics more traditionally classified as advanced calculus or elementary analysis.

A number of the paradoxes touch upon classical examples, specialized functions, or canonical curves. In such instances, we mention the paradoxes or specific examples by name, in either the initial presentation or the solution of the paradox, whichever place seems more fitting. If the nature of the paradox depends on the definition of the precise curve, such as in the paradoxes involving the Koch snowflake or the Sierpinski carpet, we will include the curve name or paradox name at the outset. On the other hand, at times where we prefer to keep the option of open-ended discovery available, we may not provide the classical terminology until the solution is presented. When possible, we have also included references that treat related material in an expository or introductory manner. In this way, we hope to offer readers resources for projects, classroom presentations, or subsequent inquiry. Much of the book's content can be viewed as recreational mathematics and can be used as a natural stepping stone to further investigations into mathematical topics.

The word *sophism* comes from the Greek word *Sophos* which means *wisdom*. In modern usage it denotes intentionally invalid reasoning that looks formally correct, but in fact contains a subtle mistake or flaw. In other words, a sophism is a false proof of an incorrect statement. Each such "proof" contains some sort of error in reasoning. Plato in his desire to pursue the truth found nothing more deplorable than Sophists using deliberately deceptive arguments for personal empowerment. Priestley [29, pp. 75–80] provides a pleasant overview of this history in *Calculus: A Liberal Art*.

Many students are exposed to sophisms at school. The exercise of finding and analyzing the mistake in a sophism often provides a deeper understanding than a mere recipe-based approach in solving problems. Typical algebraic examples of flawed reasoning that produce sophisms include division by zero or taking only a nonnegative square root. The following sophism from basic algebra utilizes the trick of division by zero to "prove" that "$1 = 2$".

Statement: If $x = y$, then $1 = 2$.

Proof:

$$x = y$$

Thus

$$xy = y^2,$$

And

$$xy - x^2 = y^2 - x^2,$$

so that

$$x(y - x) = (y + x)(y - x).$$

Now dividing by $y - x$ gives

$$x = y + x.$$

Then substituting $x = y$ on the right gives

$$x = 2x.$$

Finally dividing by x gives

$$1 = 2.$$

In this book the tricks or incorrect reasoning steps that lead to the sophisms are tied to calculus concepts. The examples are designed to reinforce the correct understanding of oft-misconstrued principles. According to our usage of the terms paradox and sophism, many well-known so-called paradoxes, such as Zeno's paradoxes and Aristotle's wheel paradox will be viewed as sophisms in this book. When classical sophisms are treated, we

provide references when possible. We hope that interested readers will use the ideas in this book as a springboard for future mathematical investigations.

In a manner similar to the first author's previous book, *Counterexamples in Calculus* [17] this book aspires to encourage students and teachers to examine paradoxes and sophisms that arise in calculus for these purposes:

- To provide deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance mathematical thinking beyond algorithmic or procedural reasoning
- To enhance baseline critical thinking skills—analyzing, justifying, verifying, and checking
- To expand the example set of noteworthy mathematical ideas
- To engage students in more active and creative learning
- To encourage further investigation of mathematical topics

In that regard, this book may well serve

- High school teachers and university faculty as a teaching resource
- High school and college students as a learning resource for calculus
- Calculus instructors as a professional development resource

Contact Details for Feedback

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