Introduction

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Section 1 of this introduction explains the rationale for this book. Section 2 discusses what we chose *not* to include, and why. Sections 3 and 4 contain a brief summary of historical background leading to contemporary perspectives in the philosophy of mathematics. Section 3 traces the history of the philosophy of mathematics through Kant, and Section 4 consists of an overview of the foundational schools. Section 5 is an annotated bibliography of sources for interesting recent work by some influential scholars who did not write chapters for this book. And finally, section 6 consists of very brief overviews of the chapters in this book.

1 The Purpose of This Book

This book provides a sampler of current topics in the philosophy of mathematics. It contains original articles by leading mathematicians, mathematics educators, and philosophers of mathematics written with a mathematical audience in mind. The chapters by philosophers have been edited carefully to minimize philosophical jargon, and summarize many years of work on these topics. They should thus provide a much gentler introduction to what philosophers have been discussing over the last 30 years than will be found in a typical book written by them for other philosophers. We have also included a glossary of the more common philosophical terms (such as epistemology, ontology, etc.). The chapters by mathematicians and mathematics educators raise and discuss questions not currently being considered by philosophers.

The philosophy of mathematics, starting about 1975, has been undergoing something of a renaissance among philosophers. Interest in foundational issues began receding and philosophers returned to more traditional philosophical problems. Meanwhile, some developments in mathematics, many related to the use of computers, have reawakened an interest in philosophical issues among mathematicians. Yet there is no book on these issues suitable for use in a course in the philosophy of mathematics for upper-level mathematics majors or mathematics graduate students, or for mathematicians interested in an introduction to this work. (Hersh's recent collection [Hersh 2005] contains many interesting articles related to the philosophy and sociology of mathematics, and is accessible to a similar audience, but it does not attempt, as we do, to cover the range of current discussion in the philosophy of mathematics.)

Our principal aim with this volume is to increase the level of interest among mathematicians in the philosophy of mathematics. Mathematicians who have been thinking about the philosophy of mathematics are likely to enjoy the variety of views in these papers presented in such an accessible form. Mathematicians who have never thought about philosophical issues but wonder what the major issues are should find several chapters to whet their interest. Those teaching courses in the philosophy of mathematics for upper-level mathematics undergraduates (or others with a similar mathematical background) should find it a useful collection of readings to supplement books on the foundational issues. Moreover, we hope to encourage more dialogue between two communities: mathematicians who are interested in the philosophy of mathematics, and philosophers who work in this field. We expect that most readers will not read every chapter in this book, but will find at least half to be interesting and worth reading.

2 What is not Included in This Book

A few words about our selection of topics for inclusion in this book are in order. We have not tried to include every topic that has ever been discussed in the philosophy of mathematics, or even everything currently being worked on. In part because we do not have adequate expertise to edit such articles, we have not included anything on the philosophy of statistics, which is currently a quite active field (although we do have a chapter on the philosophy of probability). More importantly, we have chosen *not* to include articles on the three foundational schools that developed in the late 19th and early 20th centuries: logicism, intuitionism, and formalism. They are described briefly later in this introduction, and much more thorough accounts of them appear in many books, including Stephan Körner's *The Philosophy of Mathematics*, Alexander George and Daniel Velleman's *Philosophies of Mathematics*, Marcus Giaquinto's *The Search for Certainty*, and Dennis Hesseling's *Gnomes in the Fog*. While there is still active work continuing in these fields, in our view the century from approximately 1865 to 1965 was an anomalous one for the philosophy of mathematics. What had seemed, prior to this period, to be the most certain form of human knowledge, mathematics, suddenly appeared to rest on shaky foundations. Thus essentially all work in the philosophy of mathematics during this period focused on trying to determine what basis we have for believing mathematical results. Gradually, problems were found with each of the foundationalist schools. Meanwhile new paradoxes did not appear despite an enormous growth in mathematics itself. As a result, the concern about mathematical coherence decreased, and philosophical attention began to return to more traditional philosophical questions. This book, then, concentrates on this new work, and complements the four books, just mentioned, that quite adequately discuss this foundational work.

Today there are many philosophers actively working in the philosophy of mathematics. A number of the better-known among them were invited to contribute to this book. Some of them declined due to prior writing commitments. However, several very well respected philosophers of mathematics *have* written chapters for this volume, and other viewpoints are well represented by some younger philosophers who were recommended by their mentors. Thus, most current viewpoints in philosophy are represented here. However, a single volume cannot hope to do this in full detail.

3 A Brief History of The Philosophy of Mathematics to About 1850

Although this book is concerned with recent developments in the philosophy of mathematics, it is important to set this work in the context of previous work. Thus I have written this historical section despite little expertise in the subject. Much of this material comes from, and is discussed in more detail in, chapter 1 of [Körner 1968]. Moreover, I am grateful to Charles Chihara for his many suggestions on how to improve my first version of this section. Any errors that remain here are my responsibility, not his.

The way a culture approaches mathematics and its use directly influences its philosophy of mathematics. Mathematics has been of interest to philosophers at least since ancient Greece. It has been used primarily as a touchstone to explore and test theories of knowledge. Traditionally, knowledge comes from two sources: sense perception and human reasoning. Mathematical knowledge has generally been taken as the archetypical example of the latter.

Plato is particularly important to any understanding of the history of the philosophy of mathematics, for two reasons. First, he is the earliest known philosopher who saw mathematics (which, for him, was synonymous with geometry) as central to his philosophical discussions. Ancient texts assert he viewed mathematics as so important that above the door of his Academy, Plato inscribed "Let no one who is not a geometer [or, "who cannot think geometrically"] enter." Plato used mathematical examples throughout his dialogues for various purposes. For example, in *Meno*, there is a famous sub-dialogue between Socrates and a slave boy. In it, Socrates leads the slave boy to discover that if you want to double the area of a square, you must take a square whose side is the diagonal of the original square. This discussion is used to explore an idea Plato wants to propose, of knowledge as memory from a previous life. There are several excellent books on Plato's philosophy of mathematics and the mathematics of Plato's time: for example, [Brumbaugh 1954] and [Fowler 1999].

Second, some of Plato's general philosophical views have resulted in his name being given to what is still seen, by philosophers today, as the default philosophy of mathematics, "platonism." That is, "platonism" is the view that (1) there are mathematical objects, (2) these are abstract objects, existing somewhere outside of space and time, (3) mathematical objects have always existed and are entirely independent of people, (4) mathematical objects do not interact with the physical world in any "causal" way—we cannot change them, nor can they change us—and yet, (5) we somehow are able to gain knowledge of them. These properties come from Plato's theory of "Forms," which appears in his later dialogues, primarily the *Republic* and *Parmenides*. Plato was struggling with our everyday world of appearance, trying to discern what is permanent and dependably true. This led him to the idea of the form of an object (say, a table) as a sort of ideal limit toward which objects of the physical world are striving but are imperfect copies. In this realm of forms live the assorted mathematical objects we work with: numbers, geometric objects, and so on. Objects in the realm of the forms are apprehended by reason, rather than by the senses. The appeal of viewing geometric objects, so central to Greek mathematics, this way is apparent. We see imperfect lines and points in the physical world and can easily imagine a perfect point and line. Mathematical statements are necessarily true, because they describe objects in this unchangeable realm. Objects in the physical world "participate in" the forms that describe them, and, because they are only imperfect likenesses, are only approximately described by mathematical theorems.

Aristotle objected to abstracting properties of objects into an independent existence. Rather, you can discuss these abstracted *properties*, but they reside in the objects they're abstracted from. Mathematical statements are then idealizations of statements about objects in the physical world. To the extent that these idealizations are accurate representations of the physical objects they're abstracted from, mathematical theorems can be approximately applied to physical objects. Two other contributions Aristotle made to the philosophy of mathematics were a discussion of infinity, and the beginnings of logic. Aristotle's distinction between potential infinities (basically, what happens when we take the limit as $x \to \infty$) and actual infinities (such as the set of integers, real numbers, etc.) was important historically in mathematicians' hesitation to accept many developments involving actual infinities.

Gottfried Wilhelm Leibniz was, of course, one of the founders of calculus, but he also made a substantial contribution to logic and to philosophy. He believed that by developing a systematic calculational logic (a "calculus ratiocinator"), one could represent much human reasoning and resolve many differences of opinion. He began to develop such a system, and introduced many of the modern logical concepts: conjunction, disjunction, negation, etc. (None of this, however, was published during his lifetime.) For Leibniz, mathematical facts are truths of reason, "necessary" truths whose denial is impossible (as opposed to truths of fact, that are "contingent," that just happen to be true in this world, and whose denial is possible). Mathematical facts are true in "all possible worlds" (a terminology he introduced).

John Stuart Mill was a complete empiricist about mathematics as about everything else. He believed that mathematical concepts are derived from experience and that mathematical truths are really inductive generalizations from experience. There are no necessary truths. Thus every mathematical theorem can, in principle, be found to be false and in need of revision. Mathematical truths are about ordinary physical objects. Geometrical propositions are inductively derived from our experience with space, and are taken to mean that, the more closely physical objects approach these idealized geometrical objects, the more accurately the theorems can be applied to them. Statements such as "$2 + 3 = 5$" is a generalization about how many objects you get when you put together a pile of two and a pile of three physical objects. However, for people, such inductive generalizations are a psychological necessity, because they come from very deep and invariant experiences. These experiences create an appearance of mathematical facts being necessarily true.

For **Immanuel Kant**, mathematics provided central examples for his classification of knowledge. Knowledge of propositions was classified into *a priori* or *a posteriori*. Meanwhile, propositions were classified as synthetic or analytic. A proposition is known *a priori* ("from the former"—before experience) if it is known without any particular experiences, simply by thinking about it. A proposition is known *a posteriori* if knowledge of it is gained from experience or via the senses. An "analytic" proposition is one whose predicate is contained in its subject. For example, "all squares are rectangles" is analytic because the definition of square (as "a rectangle with congruent sides") contains the requirement that it be a rectangle. The canonical example of an analytic proposition is "all bachelors are unmarried." Many mathematical and logical truths are analytic and are known *a priori*, as with "all squares are rectangles." A proposition that is not analytic is "synthetic." According to Kant, most truths about the world—"Mount Everest is the highest mountain in the world," for example—are synthetic, and are known *a posteriori*. It is generally believed that no analytic propositions can be known *a posteriori* (although a modern philosopher, Saul Kripke, has disputed this).

This leaves the category of synthetic propositions that are known *a priori*. According to Kant, our intuitions of time and space, which give us facts about the real numbers ($\mathbb{R}^3$ in particular) and the integers (such as $2 + 5 = 7$), are synthetic, yet are known *a priori*. We do not get them simply by analyzing their definitions, but rather by thinking about space and time. (Frege disagreed with this, at least in the case of arithmetic facts; he viewed them as analytic, and this was part of the point of his *Foundations of Arithmetic*.) Nonetheless, actual experience with space or time is not required to get this knowledge. In Kant's case, by space, he meant Euclidean space; that is, Euclidean geometry gives us our intuition of space. Thus Euclidean geometry is the inevitable necessity of thought, rather than being of empirical origin. The integers come from our intuition of time in the form of one moment, then the next moment, and so on. (This idea first appeared much earlier, in Plato's *Timaeus* 39b-c.) Another distinction Kant makes is between concepts we can both perceive and construct—such as the concept of two objects—those we can construct but not perceive—such as $10^{10^{10^{10}}}$—and those we can neither construct nor perceive, but are simply "ideas of reason" because they are consistent—such as actual infinities.

4 The Foundational Problems and the Three Foundational Schools

In the nineteenth century, three events occurred that caused both mathematicians and philosophers to reassess their views of issues such as what mathematics is about, how we acquire mathematical knowledge, and how mathematics can be applied to the physical world.

First came the inconsistencies in the use of limits in calculus. Soon after the introduction of calculus, there were concerns about foundational issues. Derivatives were found by taking a ratio of two infinitesimal quantities and then treating the denominator as if it were zero (even though if the denominator *is* zero, one could not even have formed the quotient). Bishop George Berkeley, in *The Analyst*, 1734, inveighed against "ghosts of departed quantities." Maclaurin responded by showing how one can derive calculus results via contradiction in the "manner of the ancients," the method of exhaustion—thus, calculus is simply a short-cut to legitimate results. Lagrange (1797) responded to the problem by trying to use power series to get rid of limits. This introduced its own problems, in the absence of some way of seriously considering issues such as divergence. For example, Grandi, in 1703 (see [Burton 2003], p. 567) set $x = 1$ in

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{to get} \quad \frac{1}{2} = 0.$$

These difficulties were of increasing concern to mathematicians in the 1800s, and the need for developing textbooks for university students eventually led Cauchy, in the 1820s, to give careful definitions of continuity, differentiability, and the integral. But Cauchy did not notice the need for a distinction between pointwise and uniform convergence of functions. As a result, he stated a false theorem about convergence of sums of continuous functions. This led to the development of careful definitions of the limits by Weierstrass and of the real numbers by Dedekind. Fourier series introduced a new set of complications: when did they converge to a function, and what exactly was a function anyway? More broadly, mathematicians began to be concerned about the foundations of mathematical beliefs—how can we be sure that what we develop is free of contradictions? This concern led to logicism, the attempt to reduce all of mathematics to logic. This work began with Gottlob Frege's *Foundations of Arithmetic* in 1884. In his 1879 *Begriffsschrift*, he developed the first fully formalized axiomatic development of the sentential calculus; he also introduced quantifiers and expanded his calculus to a predicate calculus. This was the basis for modern predicate logic, a major advance over the Aristotelian logic that had dominated for centuries. It provided him the language to formalize arithmetic. It was hoped that if there were any contradictions in mathematics, they would inevitably be found before they could cause any damage once everything was reduced to logic. One thing this formalization of arithmetic accomplished was to make statements such as $2 + 5 = 7$ a consequence of the definitions of the numbers involved, and thus turn them into analytic propositions.

The second event was the development of non-Euclidean geometries: Lobachevsky (lectured on in 1826, published about 1835), Bolyai (published in 1831 as an appendix to a book by his father), Gauss (who apparently discovered it earlier but did not publish it), and later Riemann (1854). Apparently because of the obscurity of the journals in which the work of Lobachevsky and Bolyai published, it was not until Riemann's work that the world-view of Kant was finally rejected. (Kant's world-view (see section 3, above) was that Euclidean space is the inevitable necessity of thought, rather than being of empirical origin.) Euclidean geometry was no longer the science of space—it is still far from clear which geometry is best to describe actual physical space. This revolutionary development threw mathematics out of the physical world (though, of course, not out of its usefulness in describing that world). It also led to the use of axiomatics as a way of discovering new mathematics.

The third event causing a revolution in the philosophy of mathematics was the discovery of contradictions (paradoxes) in naïve set theory. These contradictions were discovered not only for the work of Cantor (which many mathematicians were already suspicious of, as it dealt with "completed infinities") but right there in the careful work of Frege. Frege's *Grundgesetze der Arithmetik* (Fundamental Laws of Arithmetic) is a work of logicism reducing the truths of arithmetic to theorems of logic. The second volume was at the publisher in 1903 when Russell wrote to Frege, informing him of the inconsistency of his system via the Russell paradox. (The set consisting of all sets that are not members of themselves both *must be* and *must not be* a member of itself. That is, let $A = \{B : B \text{ is a set and } B \notin B\}$. Then both $A \in A$ and $A \notin A$ lead to contradictions.) Thus, the elementary step of forming the set of all objects having a given property can lead to a contradiction. Since mathematicians frequently form sets this way, this discovery shook a larger portion of the mathematical world than the others. Instead of the occasional misuses of limits, which were viewed as the result of bad mathematical taste, the view now was that there was a *crisis in the foundations of mathematics*. Was all of mathematics a house built on shifting sands? In response to this crisis, two additional foundational schools, intuitionism and formalism, were developed. Logicism was also further developed, by Russell and Whitehead, Zermelo, and others, trying to mend the problems in Frege's account.

Logicism is the thesis that mathematics is a sub-branch of logic, that all theorems of mathematics can be reduced to theorems of logic. Logic had experienced significant development in the nineteenth century in the work of Boole, De Morgan, C. S. Peirce, and Venn, among others, as well as, of course, Frege. This work made logic a very systematic study of correct rules for reasoning. Therefore, it seemed plausible that if all of mathematics could be deduced from logic, mathematics would be free of contradictions and its foundations firm. In addition, the work of Peano giving axioms for the natural numbers, of Dedekind building the real numbers, and of Weierstrass defining limits, gave logicists much material needed to reduce mathematics to logic. Frege began this work with his *Grundlagen der Arithmetik* (Foundations of Arithmetic) and continued it with his two-volume *Grundgesetze der Arithmetik* (Fundamental Laws of Arithmetic). However, the Russell paradox meant that a different approach needed to be taken. Russell and Whitehead developed one such approach in *Principia Mathematica*, an enormous work comprising three volumes and over 2000 pages. Their hope was to show in this work that all of mathematics (or at least, number theory) could be reduced to logic. Russell had analyzed the Russell paradox and other paradoxes of set theory, and determined that all of them involved defining a set by using a larger set of which the set being defined was a member, which he called a "vicious circle." He believed that, as long as one avoided using vicious-circle definitions (also called "impredicative definitions"), one could avoid paradoxes. To do this, the set theory developed in *Principia Mathematica* builds sets in a hierarchy, a type-theory, with sets of the lowest type being individuals. On the next level are sets composed of these individuals. At each level, sets are built up of members that are sets from previous levels. The known large sets that lead to contradictions cannot be constructed in this system. (For brevity, the description here is significantly simplified. Their actual approach used propositional functions, rather than sets, as the basic objects on which everything else was built, and a "ramified" theory of types. See *Ontology and the Vicious-Circle Principle* [Chihara 1973], chapter 1, for a good description.)

Logicism in this revised form had three significant problems, which largely led mathematicians to lose interest in it. First, with a rigid type-theory, many important mathematical theorems not only cannot be proven, they cannot even be stated. For example, the least upper bound of a bounded set of real numbers is defined in terms of the set of real numbers. Therefore it must be of a higher type than the real numbers. Thus, this least upper bound cannot be, in this type-theory, a real number. To overcome this problem, an additional axiom was added to their system, called the axiom of reducibility. This axiom essentially says that a set that is defined at a higher level using only sets at some lower level is equivalent to some set that appears at the first level above all those involved in its definition. The problem with this axiom is that there is no justification for it within logic (and there are some concerns that it might allow the paradoxes to reappear). Hence, the program of reducing mathematics to logic fails: either you cannot get well-known theorems, or you must add a principle that is not purely logical.

There is a similar problem with the axiom of infinity. For much of mathematics, we need infinite sets. Yet their existence simply does not follow from other axioms. Russell and Whitehead introduced it as an axiom, but cannot justify it based purely on logic. Later logicians have attempted to overcome these two problems, most notably Quine, but no one has managed to build up all of mathematics purely from logical principles.

Third, Gödel's incompleteness theorem dealt a very significant blow to even the *possibility* of deriving all of mathematics from logic. At least for consistent first-order, recursive axiomatizations of number theory, this theorem says that if they are sufficiently strong to prove normal arithmetic properties, then there are theorems that are true but not provable in such systems. Hence, one simply cannot get all of mathematics from (at least first-order) logic.

Intuitionism is the thesis that mathematical knowledge comes from constructing mathematical objects within human intuition. Intuitionism's ancestors were Kant, Kronecker, and Poincaré. Kant contributed an intuition of the integers from our *a priori* intuition of time. Kronecker was famous for his statement "God made the natural numbers; all else is the work of man." He objected to any mathematical object that could not be constructed in a finite way. In particular, he fought Cantor's transfinite numbers. Poincaré viewed logic as sterile, and set theory as a disease. On the other hand, he viewed mathematical induction as a pure intuition of mathematical reasoning.

L.E.J. Brouwer, the founder of intuitionism, believed that the contradictions of set theory came from inappropriate dependence on formal properties, including logic. In particular, the use of the law of the excluded middle (that either a statement P or its negation $\sim$P must be true) with completed infinities or with proofs of existence is illegitimate and dangerous to the coherence of mathematics. Brouwer started with Kant's idea that our intuition of time is the basis for the natural numbers. Mathematical objects are mental constructions, which Brouwer described as "intuited non-perceptual objects and constructions which are introspectively self-evident." ([Körner 1968], p. 120) Completed infinities cannot be inspected or introspected, and so are not part of mathematics. A mathematical statement is true "only when a certain self-evident construction had been effected in a finite number of steps." ([Burton 2003], p. 661) To prove a proposition of the form "P or Q," one needs to prove P or to prove Q. To prove that $(\exists x)\mathrm{P}(x)$, one needs to give a construction of an object and show that it satisfies P.

The rejection of completed infinities causes problems in the construction of real numbers. To define a real number, the intuitionist must, for example, give an algorithm that produces a sequence of rational numbers and give a proof that that sequence converges.

Many standard theorems are not intuitionistically true. For example, the standard proof of the Intermediate Value Theorem involves repeatedly bisecting an interval on which the function changes from being below the desired value C to being above it (or vice versa), maintaining that property in the sub-interval chosen. However, intuitionistically, one cannot always *determine* whether a given real number is greater than, equal to, or less than C. For example, let

$$a_n = \begin{cases} 1 & \text{if } 2n \text{ is the first even integer that is not the sum of two primes, } n > 1,\ n \text{ even} \\ -1 & \text{if } 2n \text{ is the first even integer that is not the sum of two primes, } n > 1,\ n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Define the real number $r = \sum_{n=2}^{\infty} a_n 10^{-n}$. Both intuitionistically and classically, r is a well-defined real number: to calculate its nth digit, just check if all even integers from 4 to $2n$ can be written as the sum of two primes. If the Goldbach conjecture is true, $r = 0$. If it is false and first fails at a multiple of 4 (i.e., n as used above is even), $r > 0$. If it first fails at an integer congruent to 2 modulo 4, $r < 0$. You can calculate r to whatever degree of accuracy you wish, simply by trying to decompose the appropriate values of n into sums of two primes. But while, classically, r must be either positive, negative, or zero, intuitionistically it is none of these until we decide the Goldbach conjecture. One can easily use r to give a function that shows that the Intermediate Value Theorem is not true intuitionistically, not just that there is a problem with the usual proof.

Intuitionism was developed in the same period that many abstract areas of modern mathematics—topology, functional analysis, etc.—were being developed. Most mathematicians were more interested in exploring these new developments than in retreating inside the shell of intuitionism.

There are many philosophical problems with intuitionism as well. If mathematical objects are mental constructions, there is no good reason to believe that two people will construct the same objects or have the same theorems. It is also not clear why mathematics is so useful in the world. Furthermore, much of modern physics uses mathematical objects (from functional analysis, for example) that intuitionists do not accept.

Intuitionism initially received enthusiastic support from Hermann Weyl (although he fell away from it later). Arend Heyting extended Brouwer's work in intuitionism and made Brouwer's often mystical and obscure writing much more accessible. However, because so many theorems of standard mathematics cannot be proven intuitionistically, very few mathematicians were inclined to adopt intuitionism. It required giving up too much mathematics just to avoid a few contradictions with extremely huge "sets." In the 1960s, Errett Bishop developed a variation on intuitionism, which he called constructivism (see [Bishop 1967]). He developed many theorems that are, using classical logic, equivalent to standard theorems but are constructively true. Thus, at least in analysis, one needs to give up less mathematics to be a constructivist than to be an intuitionist. This led to some renewed interest in the subject, but still has not led very many mathematicians to abandon classical mathematics.

Formalism is less well-defined. It is not clear that many serious mathematicians ever asserted the most extreme version of what is called formalism, that mathematics is just a formal game.[1] This view of mathematics is extremely unhelpful philosophically: it does not explain why we choose the axioms we choose, why mathematics is applicable to the world, why anyone would bother studying mathematics at all.

This extreme characterization of formalism appears to come from combining two parts of Hilbert's work. In his Foundations of Geometry (*Grundlagen der Geometrie*), he fixes some incompletenesses in Euclidean geometry, adopting an axiom system based on three undefined objects—points, lines, and planes—and three undefined relations—incidence (a point lying on a line), order (betweenness), and congruence. He makes it clear that, while the intuition behind the axioms comes from what we call points, lines, and planes, they could just as well stand for any objects—say, tables, chairs, beer mugs—as long as those objects satisfy the axioms. This work of Hilbert is one of the early works of modern mathematics, where, instead of working entirely within one mathematical structure, one sets up definitions and axioms and then proves theorems about the whole class of objects that satisfy the definitions and axioms.

Hilbert's proof of the consistency of his axioms for geometry reduces the question of the consistency of those axioms to the consistency of arithmetic. This brings us to the second part of Hilbert's work that is relevant for formalism. This is the "Hilbert program," aimed at restoring confidence in mathematics after the contradictions, described above, that came from work in the foundations of analysis and from naïve set theory. In part, his program was a reaction to what he considered the pernicious affect that intuitionism was having on mathematicians. He was determined to put mathematics on a sound footing without giving up large parts of mathematics in the process. The program is to first set up each field of mathematics as a formal theory, consisting of undefined terms and axioms. A proof in such a theory is a finite sequence of formulas, each of which is either an axiom or follows from earlier formulas by finitary logical rules of inference. One then investigates several metamathematical questions about the systems thus developed.

First, is the theory consistent? This can be investigated in one of two ways. One is to give a model of the theory. Usually this involves picking an already known mathematical structure (such as the integers). Then one interprets each of the undefined terms of the theory as objects within that structure in such a way that all of the axioms can be shown to be true theorems about the structure. When the structure involved is infinite, this then reduces the consistency of the original theory to the question of whether the axioms for the structure used to interpret the theory are consistent. Thus it is called a "relative consistency proof." Hilbert (in his *Grundlagen der Geometrie*) had given such proofs for Euclidean and non-Euclidean geometry by interpreting them within the real algebraic numbers. Thus, as long as the arithmetic of the real numbers is consistent, so is both Euclidean and non-Euclidean geometry. But this kind of consistency proof

[1] An exception is apparently von Neumann, who allegedly said "We must regard classical mathematics as a combinatorial game played with the primitive symbols . . . " [von Neumann 1966, pp. 50–51]. There is a quotation floating around, attributed to Hilbert: "Mathematics is a game played according to certain simple rules with meaningless marks on paper." This quotation appears for the first time in E.T. Bell, without citation—it may well have been made up by Bell. In fact, Hilbert, in [Hilbert 1919, p. 19], said "Mathematics is **not** like a game in which the problems are determined by arbitrarily invented rules. Rather, it is a conceptual system of inner necessity that can only be what it is and not otherwise." (translated by Michael Detlefsen, emphasis mine).

does not rule out the possibility that all of the theories involved are inconsistent. In addition, it is not using strictly finitary reasoning, and thus does not provide the foundation that is needed.

The second way consistency can be investigated is to show, in a finitary way, that it is not possible to derive a contradiction (for example, the statement $0 = 1$) from the axioms. This would then be an *absolute* consistency proof. It would not depend on another system (except, of course, the logic involved, which is finitary and might be acceptable to intuitionists). Of course, if one could give this kind of consistency proof for arithmetic, it would provide an absolute proof of the consistency of geometry, since a relative consistency proof had reduced the consistency of geometry to that of arithmetic.

Second, is the theory complete? This has a syntactic and a semantic meaning. Semantically, can all truths about the structure involved be proven from the axioms? If an axiomatization is not complete, then it has not captured all the relevant features of the mathematical structure it is axiomatizing, and there is a need to find further axioms so as to fully represent the structure. Syntactically, if an axiomatization is complete, every sentence or its negation is derivable from the axioms (since every sentence is either true or its negation is true in the structure).

Third, are the axioms independent of each other, or can some be eliminated? One usually shows independence by giving a structure in which all but one of the axioms are true, and the remaining one fails. This is the least important question, more an aesthetic issue than one central to the adequacy of the theory. But as mathematicians tend to like clean results, it is preferable to find axioms that are independent. Hilbert, in his *Grundlagen der Geometrie*, showed that many of his axioms were independent, though, given the tediousness of going through all combinations, he did not show that all were.

The foundational school called formalism contains as its core the view that to set mathematics on firm foundations, one should investigate these questions for the various structures and theories that make up mathematics. This led to the development of the field called proof theory, which investigates these metamathematical questions.

Unfortunately for Hilbert's program, two results of Gödel showed that the program could not work. His first incompleteness theorem showed that any consistent first-order axiomatization for the natural numbers that can be described recursively (basically, in a finitist way), and that is sufficiently strong to prove most of the standard theorems of number theory, is incomplete. That is, there are truths about arithmetic that cannot be proven within that axiomatization. (Actually, the result Gödel proved required a little more, called ω-consistency; the result was improved by Barclay Rosser to simply require standard consistency.) Thus, one cannot capture all truths about the integers within a finitistic system. His second incompleteness theorem was even more devastating. Given any consistent, recursive system of (first-order) axioms that is sufficiently strong to do a significant amount of mathematics[2], it is impossible to prove the consistency of the system within that system. Thus, there is no point in looking for a finitary proof of consistency. There has been continuing work in proof theory investigating properties of axiom systems, but there does not appear to be any hope of reviving Hilbert's original program. Gödel proved a third important theorem relevant to the Hilbert program, the completeness theorem for first-order

[2] Here, "sufficiently strong" represents a technical requirement involving being able to represent the primitive recursive functions within it and derive some standard number theoretical results; for details, see any standard textbook on mathematical logic.

logic. This says that every consistent set of first-order statements has a model. That is, our system of first-order logic is complete: in it, every first-order statement which is true in every model can be proved. Thus, semantic consistency (having a model), for first-order theories, is equivalent to syntactic consistency (not being able to derive a contradiction). Second-order theories, however, may be consistent without having any models.

Of these foundational schools, only logicism can really be called a philosophy of mathematics, as the other two do not really provide answers to all of the traditional philosophical questions: "what is the nature of mathematical objects," "what is the nature of mathematical knowledge," and "how can mathematical results help us understand physical world?" Intuitionism does not answer the last; formalism does not answer the first or the third (and, because of Gödel's results, does not answer the second either). Logicism's answer to all of these questions reduces to the similar questions about logic. However, since there are serious problems in reducing mathematics to logic, logicism does not settle these questions either. But for the first three-quarters of the twentieth century, work on foundations replaced almost all other discussion about the philosophy of mathematics.

More detailed discussions of the three foundational schools can be found in [Burton 2003]; [Körner 1968] and [George/Velleman 2002] are books, aimed at the same audience as this book, devoted to a thorough discussion of these views. Also, [Giaquinto 2002] is an accessible book that gives a good discussion of what work has been done in each of these schools.

4.1 Other Philosophers in This Period

There are two philosophers who wrote a substantial amount about mathematics during this period, but were not part of any of these foundational schools. One was **Edmund Husserl**, who developed phenomenology. He had a Ph.D. in mathematics, and his *habilitation* dissertation was *On the Concept of Number* (1887), which was later expanded to *Philosophy of Arithmetic*, published in 1891. This book attempted a psychological foundation of arithmetic, and preceded his phenomenological work, which was first published in 1900 in *Logical Investigations*. Husserl also has a very fine (and influential) essay, called "The Origin of Geometry," that usually appears as an appendix to his *The Crisis of European Sciences and Transcendental Phenomenology*. Derrida's Ph.D. thesis is a response to it. Husserl is quite difficult to read. Richard Tieszen has worked on making Husserl accessible, as well as answering philosophical objections to Husserl's work; see his *Phenomenology, Logic, and the Philosophy of Mathematics* [Tieszen 2005].

Ludwig Wittgenstein is another influential philosopher of this period who is also not easy to read. His work focuses on "language games," or the relation between language, as we use it, and reality. His initial work on this topic in the *Tractatus Logico-Philosophicus* (1922) set the stage for his work on the philosophy of mathematics in *Philosophical Remarks* (1929–30), *Philosophical Grammar* (1931–33), and later in *Remarks on the Foundations of Mathematics* (1937–44). According to the Stanford Encyclopedia of Philosophy, Wittgenstein maintains that mathematical propositions differ from real propositions. Mathematical statements do not refer to anything real, but their content comes from their syntax. "On Wittgenstein's view, we invent mathematical calculi and we expand mathematics by calculation and proof, and though we learn from a proof that a theorem *can* be derived from axioms by means of certain rules in a particular way, it is *not* the case that this proof-path pre-exists our construction of it." (http//plato.stanford.edu/entries/wittgenstein-mathematics/) He views mathematics as a human invention, and no mathematics exists until we discover it. Wittgenstein is thus a precursor of some social-constructivist views of mathematics.

5 More Recent Work That is Worth Reading but is Not Represented Here

As I mentioned in the first section of this introduction, this book consists of original articles by philosophers, mathematicians, and mathematics educators, most summarizing work over a period of years. To put this book together, I invited people whose work I had read and admired to write a chapter for this volume. I got a relatively good response, and thus this volume covers a fairly wide range of contemporary issues. However, in part because I was often asking very senior people in the field, there were a number of excellent writers on the philosophy of mathematics who declined to participate in this project. You'll certainly find suggestions for continued reading on any of the topics in this book in the bibliographies of the individual chapters. However, I want to take some space here to recommend some other very good places to learn more about the philosophy of mathematics. Full bibliographic references for these books and articles are in the Bibliography at the end of this introduction.

5.1 Logicians with a Philosophical Bent

Two logicians who have done a significant amount of very thoughtful and careful work in the philosophy of mathematics have recently collected that work in books: **Solomon Feferman**'s (math.stanford.edu/~feferman/) *In the Light of Logic* [1998] and **William Tait**'s (home.uchicago.edu/~wwtx/) *The Provenance of Pure Reason: Essays in the Philosophy of Mathematics and Its History* [2005]. I recommend both books highly.

5.2 Philosophers

There are many philosophers working in the philosophy of mathematics, almost all of them working on questions of the nature of mathematical objects and of mathematical knowledge: the debate, represented and summarized in this volume by the chapters by Balaguer, Chihara, Linnebo, and Shapiro, of platonism versus nominalism. All of the philosophers listed below have written a lot more than is mentioned here, of course; but I'm pointing to those I think are likely to be interesting to mathematicians.

Paul Benacerraf, at Princeton (philosophy.princeton.edu/components/com_faculty/documents/paulbena_cv.pdf) wrote two seminal papers that initiated the move in the 1970s, by philosophers of mathematics, back to traditional philosophical questions and away from foundations: "What Numbers Could Not Be" [1965] and "Mathematical Truth" [1973]. They are still well worth reading. He also, with Hilary Putnam, edited a book of readings in the philosophy of mathematics. It has gone through two editions, with a quite different selection of papers in the second edition. Both editions are worth looking at. His two articles mentioned above are reprinted in the second edition.

John Burgess (www.princeton.edu/~jburgess/), also at Princeton, works in logic (philosophical and mathematical) and the philosophy of mathematics. What he says about mathematics is very careful and correct. However, since he works in the philosophy department, his interests have been turning more and more toward technical philosophical issues. His "Why I Am Not a Nominalist" [1983] is quite accessible. I have not read his two recent books: *A Subject with No Object* [Burgess/Rosen 1997] and *Fixing Frege* [2005]. I looked at the former and decided that it was far more technical than I could handle without devoting months to it. I do hope to look at the latter once this book is finished.

Imre Lakatos combined the approaches of philosophers of science Thomas Kuhn and Karl Popper and applied it to mathematics. His best-known work is *Proofs and Refutations* [1976], a lively dialogue about Euler's theorem that $v - e + f = 2$ (where v represents the number of vertices, e the number of edges, and f the number of faces) for a polyhedron. It shows how cases that are counterexamples motivate revisions of the hypotheses of the theorem and the definition of polyhedron. This provides, according to him, an example of how mathematics develops.

Penelope Maddy (www.lps.uci.edu/home/fac-staff/faculty/maddy/), at the University of California at Irvine, started her career as a student of Burgess and a platonist. Her first book, *Realism in Mathematics* [1990], described an unusual form of platonism in which mathematical objects are located in the physical world. This view was broadly attacked by other philosophers. Her current direction, a naturalist approach to the philosophy of mathematics (*Naturalism in Mathematics* [1997]), is one that takes science as the standard by which all knowledge is to be judged. Knowledge of anything, including mathematics, must be justifiable through our best scientific theories, in particular, empirical psychology, linguistics, etc.

Charles Parsons (www.fas.harvard.edu/∼phildept/parsons.html), at Harvard, works in the philosophy of mathematics as well as in logic and in other fields of philosophy. His article "Mathematical Intuition" [1979–80] is a fairly interesting discussion of how one can have intuitions of mathematical objects such as numbers and sets. However, it will be disappointing if you are expecting something like what Poincaré wrote on the topic. Many of his papers in the philosophy of mathematics are collected in [Parsons 1983].

Hilary Putnam (www.fas.harvard.edu/∼phildept/putnam.html), also at Harvard, works on philosophy of mathematics, philosophy of science, and other fields of philosophy. His article, "Mathematics without Foundations" [1967], is one of the early articles moving philosophy of mathematics back from foundations to more traditional philosophical problems. Some of his work in the philosophy of mathematics is collected in [Putnam 1985].

Michael Resnik (http://philosophy.unc.edu/resnik.htm), at the University of North Carolina, Chapel Hill, is a structuralist (of a slightly different sort than Stewart Shapiro, who has an article in this volume). His book, *Mathematics as a Science of Patterns* [1997], is quite readable once you are used to philosophical terminology. Given that many mathematicians assert that mathematics is the science of patterns, the book is worth reading to see how philosophers establish such an assertion.

5.3 People Working in the History and Philosophy of Mathematics

Several people work on the boundary between philosophy of mathematics and history of mathematics. One is **Kenneth Manders** (www.pitt.edu/∼philosop/people/manders.html), at the University of Pittsburgh. He is primarily a philosopher (and logician) with a strong mathematical background, but his arguments are very carefully historically based. Unfortunately, he rarely publishes. One published article is "Domain extension and the philosophy of mathematics"

[1989]. I have a very interesting preprint, “Why Apply Math?” from 1999, and another, “Euclid or Descartes: Representation and Responsiveness.” Both are very carefully and thoughtfully written, but the only way to get them is to write to him.

Another is **Leo Corry** (http://www.tau.ac.il/∼corry/), at Tel Aviv University. He is more of a historian of mathematics, but he asks philosophical questions. For example, his *Modern Algebra and the Rise of Mathematical Structures* [2004] investigates how the notions of what was meant by “algebra” and “mathematical structure” developed over the last two centuries.

A third is **Paolo Mancosu** (http://philosophy.berkeley.edu/mancosu/); see his “On Mathematical Explanation” [2000]. In addition to being interesting in itself, it mentions several other articles on this topic.

A fourth is **Howard Stein** (https://philosophy-data.uchicago.edu/index-faculty.cfm#Stein), at the University of Chicago, who works on the history and philosophy of mathematics and physics. Three articles worth looking at are “Yes, but . . . : Some Skeptical Reflections on Realism and Anti-realism” [1989], “Eudoxos and Dedekind: On the Ancient Greek Theory of Ratios and its Relation to Modern Mathematics” [1990] and (do not be put off by the title) “Logos, Logic, and Logistiké: Some Philosophical Remarks on 19th Century transformation of Mathematics” [1988].

5.4 People Working in Mathematics Education

By the very act of teaching mathematics, one takes a position on how people acquire mathematical knowledge, which has both psychological and philosophical aspects. Therefore many people whose research is in mathematics education have interesting philosophies of mathematics. I contacted several of them, and, as it turns out, the person whose work I find most interesting *has* made a contribution to this book, but several others whose work I also respect were either unwilling or unable to do so.

Ed Dubinsky (http://www.math.kent.edu/∼edd/) has worked applying an interpretation of Piaget’s work to higher-level mathematics education. He is very active (though now retired), and has gathered a large community of mathematics educators who work with him. He started the Research in Undergraduate Mathematics Education Community, which later branched into the SIGMAA on RUME. He started out as a functional analyst. In his early work in mathematics education, he used a computer-based language, ISETL, to help students understand abstract mathematical objects. For example, see his “Teaching mathematical induction, I/II” ([1986], [1989]). The basic theory that he developed, APOS theory (standing for Action, Process, Object, Schema), describes how students gradually develop more sophisticated concepts through a process of reflective abstraction. A description can be found in “A theory and practice of learning college mathematics” [1994]. It is applied to student understanding of functions in “Development of the process conception of function” [1992], and to abstract algebra in “Development of students’ understanding of cosets, normality and quotient groups” [1997]. An overall framework is given in “A framework for research and curriculum development in undergraduate mathematics education” [1996]. Dubinsky views his philosophy as inseparable from his educational theory and practice, and declined to write an article for this volume because he felt that his work already expresses his philosophical position adequately.

Paul Ernest (http://www.people.ex.ac.uk/PErnest/) received a Ph.D. in philosophy of mathematics, but spent much of his career working in mathematics education. He is the editor of the *Philosophy of Mathematics Education Journal*. He has written an article, "The Impact of Beliefs on the Teaching of Mathematics" ([Ernest 1994]; originally written in 1989), suggesting that to make significant changes in mathematics education requires changing beliefs about the nature of mathematics as well as about how it is taught and learned. This then led to a book, *The Philosophy of Mathematics Education* [Ernest 1991]. More recently, he wrote a book setting forth his philosophical views of mathematics itself, *Social Constructivism as a Philosophy of Mathematics* [Ernest 1998]. As I make clear in my review of that book [Gold 1999], I do not view it as a viable version of social constructivism, but not all mathematicians agree with me, and I encourage readers to decide for themselves.

Annie and **John Selden** are the editors of the Research Sampler on MAA Online (http://www.maa.org/t_and_l/sampler/research_sampler.html), which brings selected research in mathematics education to the attention of collegiate mathematics educators. After long careers at various universities in the U.S. and abroad, they are now Adjunct Professors of Mathematics, New Mexico State University. In their own research, they have examined students' ability to solve novel calculus problems ([1989] and [2000]), students' grasp of the logical structure of informal mathematical statements, student difficulties with proofs [2003], and are currently investigating (college) teachers' beliefs about mathematics, teaching, and learning. See also their Research Questions page (http://www.maa.org/t_and_l/sampler/rs_questions.html) on MAA Online.

David Tall (http://www.warwick.ac.uk/staff/David.Tall/) also connects philosophical views of mathematics with his educational work in significant ways. See his "Existence Statements and Constructions in Mathematics and Some Consequences to Mathematics Teaching" [Tall/Vinner 1982], and a book he edited, *Advanced Mathematical Thinking* [1991]. More recently he has looked at the mathematical world as really three different realms [2004], the first coming from our perceptions of the world and thinking about them, the second the world of symbols we use in mathematics, the third the formal axiomatic world.

5.5 Mathematicians

I had a better success rate getting mathematicians who are interested in the philosophy of mathematics to contribute to this book. One who did not was **Saunders Mac Lane**; anyone interested in the philosophy of mathematics will find his *Mathematics: Form and Function* [1986] worth reading.

Chandler Davis has a very interesting view of mathematics, coming from a materialist perspective; see his "Materialist Mathematics" [1974] and "Criticisms of the Usual Rationale for Validity in Mathematics" [1990].

Lynn Steen has written a number of articles (and books) popularizing mathematics that include a philosophical bent. See particularly "The Science of Patterns" [1988].

Ian Stewart has also written a number of popular books about mathematics that have substantial philosophical implications. One of the best in that direction is *Nature's Numbers: The Unreal Reality of Mathematics* [1995].

6 A Brief Overview of This Book

This book is not designed for a straight read from beginning to end, although some readers might choose to do that. It is meant to be dipped into as the topic and writing style appeals to you. Each chapter is self-contained and most are liberally sprinkled with references for those wanting to delve more deeply into a particular topic. We have tried to organize it somewhat by topic, but within each topic the style and point of view of the chapters can be quite different. Thus, we've tried to provide you here with a guide to the chapters. Also each chapter is preceded by a short description of it and a brief biographical sketch of the author.

For mathematicians who have some curiosity about philosophical questions regarding mathematics, but who have not read any contemporary philosophy, a good place to start might be **Philip Davis**'s chapter. He asks a question that we have all wrestled with at some point, after we have done some mathematics and are thinking of writing it up: when is a problem solved? When can we say, OK, let's wrap that up now?

6.1 Views on Mathematical Objects

Barry Mazur's chapter should also be very accessible to mathematicians without much philosophical background. He asks a question that is close to a traditional philosophical question—how can one tell when one mathematical object is really the same as another. However, he looks in a very different direction than philosophers generally do as he traces some category theory from fundamentals to propose an answer this question, with some interesting comments along the way.

The others writing about mathematical objects are all philosophers.

Stewart Shapiro gives an overview of philosophical discussions concerning mathematical objects. This culminates with his view that mathematics is the science of structures (as suggested originally by Bourbaki), or, as it is sometimes called, of patterns.

For mathematicians interested in reading about current work in the philosophy of mathematics, **Charles Chihara**'s chapter is a relatively gentle introduction to the kind of discussions philosophers have. He discusses concerns about the existence of mathematical objects. Finally he turns to how one can develop a structural account of mathematics without being committed to the actual existence of structures in either the world or some ideal platonic realm.

Mark Balaguer gives an overview of the major variations philosophers have discussed over the last thirty years on whether there are mathematical objects; if there are, what their nature is; and how we can gain mathematical knowledge. As a summary of thirty years of philosophical discussion, this chapter is quite dense. However, it very effectively and systematically summarizes the discussion from a wealth of philosophical views.

Øystein Linnebo develops a new view of mathematical objects that allows them to exist in some sense while avoiding some of the traditional objections to a platonist account of mathematical objects.

6.2 Views on Proof

Proof and its relation to mathematical knowledge is an issue that has become an active concern again thanks to computer-assisted proofs and mathematical investigations involving computers.

Michael Detlefsen discusses both the role of empirical reasoning—primarily due to the use of computers—and of formalization in mathematical proofs.

Joseph Auslander discusses the various roles proof plays in mathematics, and how standards of proof vary over time.

Jon Borwein focuses less on proof than on the development of mathematics, and the roles computers may play in that development. Necessarily this includes the role they play in developing proofs.

6.3 What is Mathematics?

Robert Thomas's chapter suggests, as a definition of mathematics, a variation on "mathematics is the science of patterns." He takes mathematics as one extreme end in the spectrum of the sciences, and suggests (read the chapter for what he means) that "mathematics is the science of relations as such."

Guershon Harel approaches the problem of "what is mathematics?" from the viewpoint of a researcher in mathematics education. He proposes an answer that includes not only the theorems, but also the tactics and conceptualizations we use.

6.4 Social Constructivism

Reuben Hersh, one of the few mathematicians to attempt to describe social constructivism in some detail, discusses mathematics and its development (or, as he phrases it, mathematics as "a living organism") as the subject of scientific investigation. This lively chapter includes a beautiful attempt to describe the feeling when an idea for solving a problem suddenly flashes into one's mind.

Julian Cole just finished his Ph.D. thesis in philosophy, working on how one can make social constructivism coherent from a philosopher's standpoint. His chapter summarizes the main points of interest to mathematicians of his work.

6.4.1 The Boundaries Between Mathematics and the Other Sciences (Physical and Social), and the Applicability of Mathematics

Mark Steiner looks at a particular aspect of this question (primarily from the standpoint of a philosopher interested in the application of mathematics to physics) related to generalizations of addition.

Keith Devlin looks at the question of what we currently call mathematics versus what we currently relegate to applied mathematics. He describes how he believes this will change.

6.5 Philosophy of Probability

Alan Hájek discusses some of the fundamental philosophical issues about the nature of probabilities. It is a very accessible chapter.

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