WILLIAM DUNHAM

# Quadrilaterally Speaking

Among the figures of plane geometry, the triangle holds a special place. Triangles are simple, basic, and unembellished, yet their geometric importance cannot be overemphasized. All have three medians, three altitudes, and one centroid, and all possess both inscribed and circumscribed circles. When two or more triangles get together, they can be the spitting image of one another (i.e., be congruent) or at least bear a strong family resemblance (i.e., be similar). And, as everyone knows, congruent and similar triangles are critical to the logical development of geometry.

But triangles have one major shortcoming when compared to their polygonal cousins: they lack diagonals. After drawing a triangle's three sides, the mathematician finds that no unconnected vertices remain, and consequently the "diagonal" of a triangle is the geometric counterpart of one hand clapping.

In order to explore the interconnection among sides and diagonals, it is necessary to step up to quadrilaterals. For the purposes of this article, all quadrilaterals will be convex, with both diagonals falling neatly inside. As denoted in Figure 1, quadrilateral $ABCD$ will have sides of length $a$, $b$, $c$, and $d$ and diagonals of length $x = AC$ and $y = BD$. (We will use $AC$ to represent both *the line segment* from $A$ to $C$ and *the length of this segment*. Which is meant should be clear from the context.)

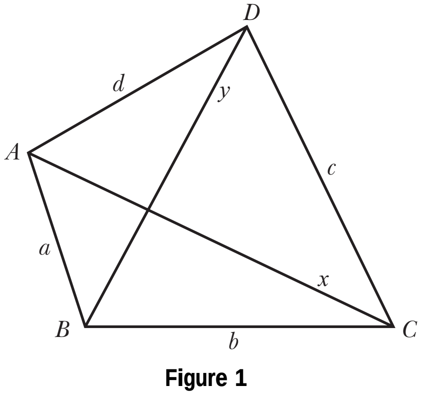


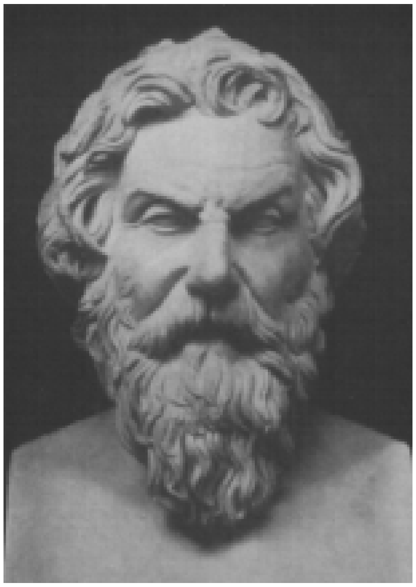

**Figure 1**

We shall examine a trio of theorems relating the sides and diagonals of quadrilaterals. Although far from new, they seem to be relatively unknown, even among the mathematically preoccupied. Subscribing to the adage that good things bear repeating, we consider them in turn.

## Ptolemy's Theorem

This result appears in Book I of the *Almagest*, the astronomical masterpiece of Claudius Ptolemy (ca. 85–165). A word of warning: the likeness we have provided of Ptolemy is not to be trusted. Indeed, it looks suspiciously like the images we have of Homer, Aeschylus, and Socrates—not to mention those of Noah, Nebuchadnezzar, and Neptune. Portraits of classical males tend to be interchangeable, with everybody more or less resembling Moses, or perhaps Charlton Heston.

But Claudius Ptolemy does hold one legitimate distinction: he is the earliest major mathematician who possessed both a first and a last name. His illustrious predecessors—from Euclid



*Claudius Ptolemy*

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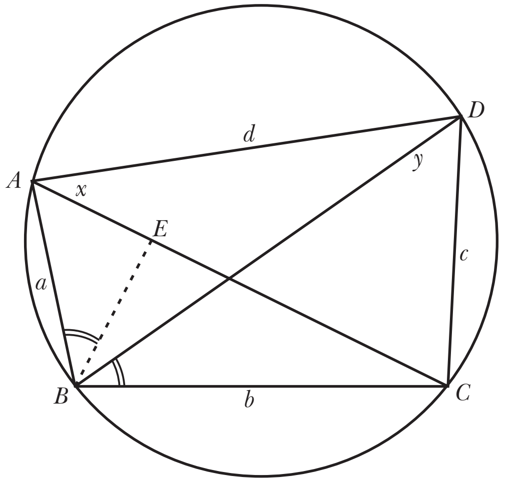


Figure 2

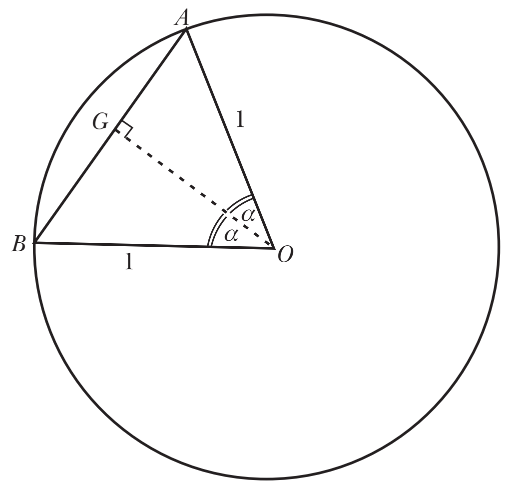


Figure 3

to Archimedes to Apollonius—got by quite well with one name. (Whether "Euclid" was his first name or his last name is a matter that concerns only chronic insomniacs.)

In any case, before stating Ptolemy's theorem, we need

**Definition:** A *cyclic* quadrilateral is one that can be inscribed in a circle.

Put another way, a quadrilateral is cyclic if a circle can be circumscribed about it. It should be clear that this is a very restrictive condition. For, suppose we refer to the quadrilateral in Figure 1 and circumscribe a circle about $\Delta ABC$ . Then quadrilateral *ABCD* is cyclic if and only if the point *D* falls exactly upon the circle just drawn— an unlikely circumstance to be sure.

In spite of the fact that cyclic quadrilaterals are far from general, Ptolemy proved a beautiful result about them, namely:

**Theorem 1.** *If ABCD is a cyclic quadrilateral, then the product of the diagonals is the sum of the products of the opposite sides. That is, in Figure 2,* $ac + bd = xy$.

*Proof.* To begin his elegant argument, Ptolemy constructed $\angle ABE$ congruent to $\angle DBC$, where *E* lies on diagonal *AC*. Because $\angle BAC$ and $\angle BDC$ intercept the same arc of the circle, they are congruent. Therefore $\Delta ABE$ is similar to $\Delta DBC$, from which it follows that $AB/AE = DB/DC$. That is, $a/AE = y/c$, or equivalently

$$ac = (AE)y \qquad (1)$$

Likewise, $\Delta ABD$ is similar to $\Delta EBC$ because $\angle ADB$ is congruent to $\angle ECB$ (they intercept the same arc) and $\angle ABD = \angle ABE + \angle EBD = \angle DBC + \angle EBD = \angle EBC$. Thus $BD/AD = BC/EC$. That is, $y/d = b/EC$, and so

$$bd = (EC)y. \qquad (2)$$

Now simply add equations (1) and (2) to get

$$ac + bd = (AE + EC)y = (AC)y = xy,$$

and the proposition is proved. QED

Ptolemy used this theorem in the *Almagest* to generate his "Table of Chords," the precursor of our trigonometric tables. Moreover, a famous identity follows as an easy corollary. To see it, we first observe that if $2\alpha$ is the measure of central angle *AOB* in the unit circle (see Figure 3) and if we draw *OG* perpendicular to chord *AB*, then $\sin\alpha = AG/OA = AG$, and so the length of chord *AB* is $2(AG) = 2\sin\alpha$.

Now for the identity. Suppose we begin with a pair of angles having measures $2\alpha$ and $2\beta$, as shown in Figure 4. Place these as central angles within a unit circle, one on either side of diameter *BD*, and connect points *A*, *B*, *C*, and *D* to generate a cyclic quadrilateral and its diagonals. By our previous observation,

$$AB = 2\sin\alpha,\ \ BC = 2\sin\beta,\ \text{ and }\ AC = 2\sin(\alpha + \beta).$$

Furthermore, because *DAB* and *DCB* are inscribed in a semicircle, the corresponding triangles $\Delta DAB$ and $\Delta DCB$ are right, and so the Pythagorean theorem implies that

$$AD = \sqrt{BD^2 - AB^2} = \sqrt{4 - 4\sin^2\alpha} = 2\sqrt{1 - \sin^2\alpha} = 2\cos\alpha$$

and similarly that $CD = 2\cos\beta$.

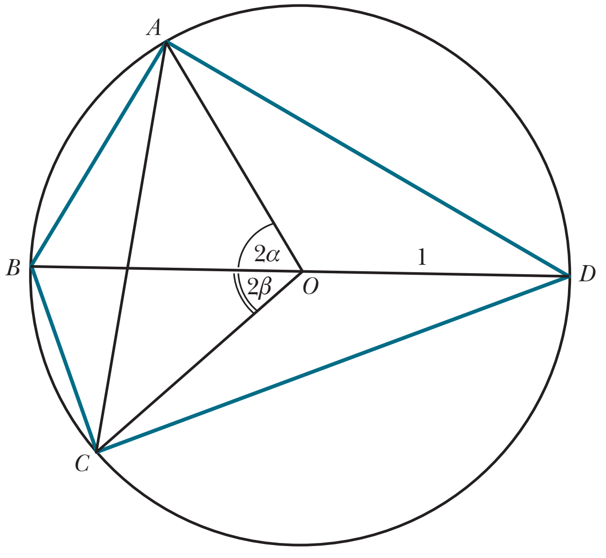


Figure 4

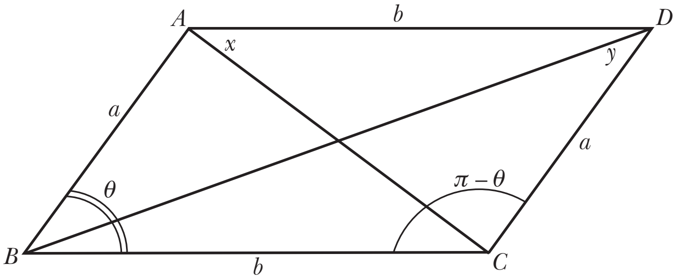


**Figure 5**

Thus when we apply Ptolemy's theorem to *ABCD*, we get

$$(2\sin\alpha)(2\cos\beta) + (2\sin\beta)(2\cos\alpha) = 2[2\sin(\alpha + \beta)],$$

which reduces to

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

the justly famous identity from trigonometry. Claudius Ptolemy was surely on to something.

Whereas Ptolemy's theorem is restricted to cyclic quadrilaterals, our second is specific to parallelograms. Yet, as we shall see, it points the way towards the general case.

**Theorem 2.** *If ABCD is a parallelogram as in Figure 5, then*

$$2a^2 + 2b^2 = x^2 + y^2.$$

Recalling that the opposite sides of a parallelogram are congruent, we can restate this as: the sum of the squares of the four sides of a parallelogram (i.e., $a^2 + b^2 + a^2 + b^2$) equals the sum of the squares of the diagonals.

*Proof.* With $\theta$ as the measure of $\angle ABC$—and thus $\pi - \theta$ as the measure of $\angle BCD$—apply the law of cosines to $\Delta ABC$ and $\Delta DBC$ to get

$$x^2 = a^2 + b^2 - 2ab\cos\theta \quad \text{and} \quad y^2 = a^2 + b^2 - 2ab\cos(\pi - \theta).$$

Then, because $\cos(\pi - \theta) = -\cos\theta$, we need only add these equations to reach the desired end. QED

We now have two theorems at our disposal—one pertaining to cyclic quadrilaterals, the other to parallelograms—and it is natural to ask under what conditions they are simultaneously applicable. That is, when is a quadrilateral cyclic *and* parallelogramic (if there is such a word)?

A moment's thought suggests that this happens if and only if the quadrilateral is a rectangle, and the proof of this double implication is easy. In one direction, we note that a cyclic quadrilateral that is also a parallelogram must satisfy both theorems 1 and 2, and so (with our prior notation)

$$xy = aa + bb = \frac{2a^2 + 2b^2}{2} = \frac{x^2 + y^2}{2}.$$

Cross multiplication yields $0 = x^2 - 2xy + y^2 = (x - y)^2$, which implies that $x = y$. But a parallelogram with equal diagonals is a rectangle, so this direction is proved. Conversely, a rectangle is already a parallelogram and is obviously cyclic, for a circle with center at the intersection of the diagonals and with radius half the length of the diagonal passes through all four vertices.

*Leonhard Euler*

In the case when *ABCD* is a rectangle with $AC = BD = x$, then theorems 1 and 2 reduce (respectively) to

$$aa + bb = x^2 \quad \text{and} \quad 2a^2 + 2b^2 = x^2 + x^2 = 2x^2.$$

In short, when both apply, they jointly collapse into the Pythagorean theorem—a most significant crossroads.

Alas, the aforementioned results hold only for special kinds of quadrilaterals. The big challenge is to prove a theorem relating the sides and diagonals of a *general* convex quadrilateral. What this relationship might be, and how to prove it, are far from obvious.

Fear not. In a 1748 paper, Leonhard Euler (1707–1783) rose to the challenge [1]. We shall present the theorem—as he did—in stages, building toward the general result after a pair of lemmas. (Note in passing that Euler looks nothing like Moses, although he does sport a natty turban.)

For the purposes of the remaining proofs, we shall explicitly draw our quadrilateral as though it is *not* a parallelogram. The reader can check that the results become trivialities in the case of parallelograms.

**Lemma 1.** *Given quadrilateral ABCD in Figure 6, complete parallelogram ABCE and draw DE. Then*

$$a^2 + b^2 + c^2 + d^2 = AC^2 + BD^2 + DE^2.$$

In words, this says that the sum of the squares of a quadrilateral's four sides equals the sum of the squares of its diagonals *plus* the square of the length of this newly created segment *DE*. Think of *DE* as measuring how far *ABCD* deviates from a parallelogram.

*Proof.* In Figure 6, construct *CF* parallel to *AD* with $CF = AD = d$ and draw segments *AF*, *BE*, *BF*, *DF*, and *EF*. It can be proved (exercise!) that quadrilaterals *ADCF* and *BDEF* are

 

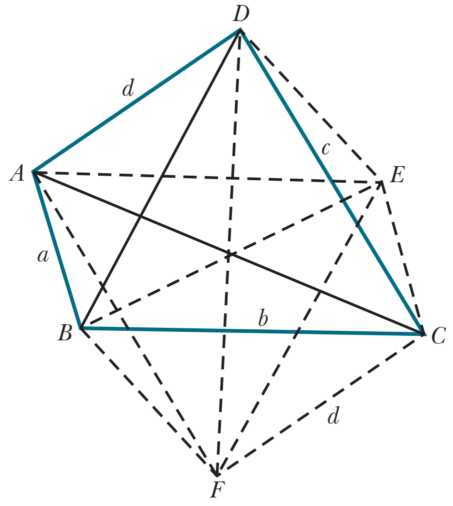


Figure 6

themselves parallelograms. Thus, by theorem 2,

$$2c^2 + 2d^2 = AC^2 + DF^2 \quad \text{and} \quad 2BD^2 + 2DE^2 = BE^2 + DF^2.$$

Consequently,

$$2c^2 + 2d^2 - AC^2 = DF^2 = 2BD^2 + 2DE^2 - BE^2,$$

and so

$$2c^2 + 2d^2 = 2BD^2 + 2DE^2 + AC^2 - BE^2.$$

But from parallelogram $ABCE$, we have $2a^2 + 2b^2 = AC^2 + BE^2$. Adding these last two equations yields

$$2a^2 + 2b^2 + 2c^2 + 2d^2 = 2AC^2 + 2BD^2 + 2DE^2,$$

and thus $a^2 + b^2 + c^2 + d^2 = AC^2 + BD^2 + DE^2$. QED

As desired, this lemma relates the four sides and two diagonals of a general quadrilateral. But it is less than optimal, requiring as it does the "extraneous" segment $DE$. To tidy up this defect, Euler pushed on:

**Lemma 2.** *For quadrilateral ABCD with completed parallelogram ABCE as above, bisect diagonal AC at P and diagonal BD at Q and draw segment PQ connecting these midpoints (see Figure 7). Then* $DE^2 = 4PQ^2$.

*Proof.* We first assert that $P$ lies on $BE$. This follows because the diagonals of parallelogram $ABCE$ bisect each other,

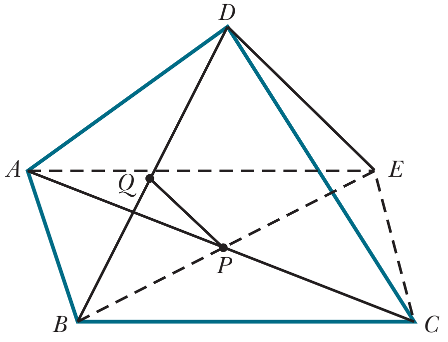


Figure 7

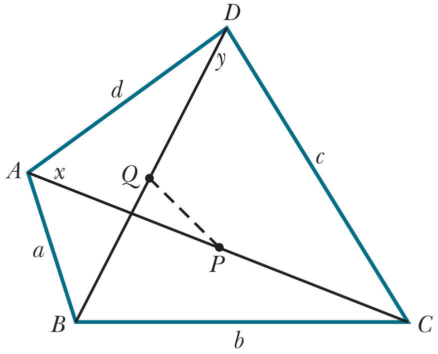


Figure 8

and so $P$—the midpoint of $AC$—must also be the midpoint of $BE$. But then segment $PQ$ connects the midpoints of sides $BE$ and $BD$ in $\Delta BED$, and by similar triangles we conclude that $2(PQ) = DE$. Squaring both sides proves the lemma.

QED

At last, we establish the general theorem (see Figure 8). Euler stated it as:

**Theorem 3.** "*In any quadrilateral, the sum of the squares of the four sides equals the sum of the squares of the diagonals plus four times the square of the line connecting the midpoints of the diagonals.*"

*Proof.* This is now trivial, for we need only combine lemma 1 and lemma 2:

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= AC^2 + BD^2 + DE^2 \\ &= AC^2 + BD^2 + 4PQ^2 \\ &= x^2 + y^2 + 4PQ^2. \end{aligned}$$

QED

Notice how the point $E$, which Euler introduced somewhat artificially at the outset, has disappeared from the final theorem—truly, Euler giveth and Euler taketh away. With this argument, he had established a curious property of quadrilaterals that was, as he put it, "neither enunciated nor proved to this point."

Euler noted a pair of interesting consequences:

> The sum of the squares of the four sides of a convex quadrilateral is always greater than or equal to the sum of the squares of its diagonals.
>
> The sum of the squares of the four sides of a convex quadrilateral *equals* the sum of the squares of its diagonals if and only if the quadrilateral is a parallelogram.

The latter follows because, under the condition of equality, the length of the segment $PQ$ is zero. Consequently, the diagonals bisect each other, and the quadrilateral is a parallelogram.

For the sake of variety, we conclude with an alternate—and shorter—demonstration of theorem 3. Although Euler needed the extraneous segment $DE$, this argument involves only the original quadrilateral. And, whereas Euler used synthetic geometry, this proof rests upon trigonometry, in particular upon a five-fold (!) application of the law of cosines.

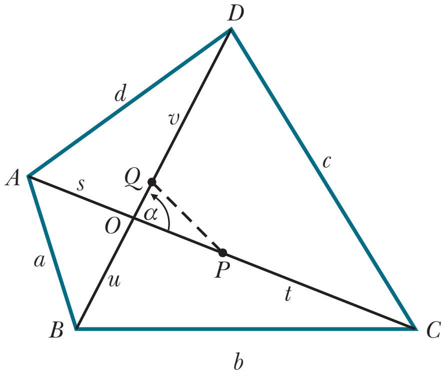


**Figure 9**

**Theorem 3 Revisited.** [2] *In any quadrilateral, the sum of the squares of the four sides equals the sum of the squares on the diagonals plus four times the square of the line connecting the midpoints of the diagonals.*

*Proof.* As illustrated in Figure 9, we again consider quadrilateral $ABCD$. With $O$ as the intersection of the diagonals, we let $OA = s$ and $OC = t$, and we stipulate without loss of generality that $s \le t$. Likewise, let $OB = u$ and $OD = v$, with $u \le v$. Finally, we let $\alpha$ be the measure of $\angle COD$. Of course it follows that $\alpha$ is also the measure of $\angle AOB$ and $\pi - \alpha$ is the measure of $\angle BOC$ and $\angle DOA$.

Apply the law of cosines individually to $\Delta AOB$, $\Delta BOC$, $\Delta COD$, and $\Delta DOA$, while remembering that $\cos(\pi - \alpha) = -\cos\alpha$:

$$a^2 = s^2 + u^2 - 2su\cos\alpha$$
$$b^2 = u^2 + t^2 + 2ut\cos\alpha$$
$$c^2 = t^2 + v^2 - 2tv\cos\alpha$$
$$d^2 = v^2 + s^2 + 2vs\cos\alpha.$$

Upon adding, we have

$$\begin{aligned} &a^2 + b^2 + c^2 + d^2 \qquad (3)\\ &= 2s^2 + 2t^2 + 2u^2 + 2v^2 - 2(su - ut + tv - vs)\cos\alpha \\ &= (s+t)^2 + (u+v)^2 + (t-s)^2 + (v-u)^2 - 2(t-s)(v-u)\cos\alpha \\ &= AC^2 + BD^2 + 4\left[\left(\frac{t-s}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 - 2\left(\frac{t-s}{2}\right)\left(\frac{v-u}{2}\right)\cos\alpha\right]. \end{aligned}$$

But recall that $P$ is the midpoint of diagonal $AC$ and $Q$ the midpoint of diagonal $BC$, as shown in Figure 9 . As a consequence,

$$OP = OC - PC = t - \frac{t+s}{2} = \frac{t-s}{2}$$

and

$$OQ = OD - QD = v - \frac{v+u}{2} = \frac{v-u}{2}.$$

Therefore, the law of cosines applied to $\Delta OPQ$ yields

$$\begin{aligned} PQ^2 &= OP^2 + OQ^2 - 2(OP)(OQ)\cos\alpha \\ &= \left(\frac{t-s}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 - 2\left(\frac{t-s}{2}\right)\left(\frac{v-u}{2}\right)\cos\alpha, \end{aligned}$$

and this last is *exactly* the expression within the square brackets of equation (3) above. A final substitution gives us, as before,

$$a^2 + b^2 + c^2 + d^2 = AC^2 + BD^2 + 4PQ^2. \qquad \text{QED}$$

Of course much more could be said about quadrilaterals. There is, for instance, the fact that if the sums of the squares of opposite sides of a quadrilateral are equal (i.e., if $a^2 + c^2 = b^2 + d^2$), then its diagonals must be perpendicular. Or there is Brahmagupta's wonderful formula giving the area of a cyclic quadrilateral in terms of the lengths of its four sides.

For now, however, we must leave this topic, reminded once again of the unexpected patterns lurking beneath the surface of Euclidean geometry. It was Howard Eves [3] who perceptively observed that this subject "though elementary, is often far from easy." As the preceding theorems reveal, quadrilaterals can hold their share of surprises.

**Note.** I wish to thank Drs. Penny Dunham and Elyn Rykken of Muhlenberg College for their helpful suggestions in the preparation of this article. ■

## References

1. Leonhard Euler, *Opera Omnia,* Ser. 1, Vol. 26, pp. 29–32.
2. Thanks to Elyn Rykken for showing me this proof.
3. Howard Eves, *A Survey of Geometry*, Allyn and Bacon, Boston, 1963, p. 64.