Figure 1. *St. Jerome in His Study*, by Albrecht Dürer, 1514.

Dürer: Disguise, Distance, Disagreements, and Diagonals!

Annalisa Crannell, Marc Frantz, and Fumiko Futamura

The year 2014 is an especially good time to tell this tale of disguise, distance, disagreements, and diagonals. It marks a five-century anniversary of a great marriage of mathematics and art. It is the tale of an artist, one of his greatest creations, and one of his detractors.

The artist we celebrate, Albrecht Dürer, was an accomplished painter and engraver. He was also a master of mathematics: He had read Euclid's *Elements,* and he wrote several influential books on geometry and proportion. In 1514, Dürer created a trio of engravings (*Knight, Death, and the Devil*; *Melencolia I*; and *St. Jerome in His Study*) that his followers have regarded as his three master prints ever since.

Many Dürer fans know of the homage to mathematics that is *Melencolia I* (see sidebar, page 11), but Dürer's perspective piece *St. Jerome in His Study* (figure 1) has some even more subtle and compelling geometry lessons on how to create—and how to look at—art.

To convince you that these lessons of *St. Jerome in His Study* are neither easy nor obvious, we'll defer to one of Dürer's detractors: William Mills Ivins Jr., curator of the department of prints at New York's Metropolitan Museum of Art from 1916 to 1946. Ivins wrote prolifically about art; several of his books remain in print today. His is an influential voice on the subject of prints, and he was convinced Dürer was in over his head when it came to perspective.

Here is a snippet from Ivins's 1938 book *On the Rationalization of Sight: de artificiali Perspectiva* [3]. On page 42 in particular, the criticism of *St. Jerome* is unequivocal and lengthy:

> If, in working out his picture, Dürer had followed the simple rules of the game as laid down by either Alberti or Viator, he would not have got himself involved in absurdity after absurdity. The top of the saint's table is of the oddest trapezoidal shape—certainly it is not rectangular. Neither is it level with the floor under it. . . . These oddities of shape were as

carefully disguised or camouflaged by shading as was possible, but anyone who cares to rule lines on a photograph or reproduction of this engraving will find these and many more to keep them company.

Was Ivins right? Did Dürer involve himself in "absurdity after absurdity"? When he drew the table, did he draw a trapezoid? Or did he draw (as we claim) a square? Is the table level with the floor?

Surprisingly, the answers to these questions depend not only on what Dürer did 500 years ago, but also on what Ivins did in 1938. And, as we will show, it depends on what *you,* the reader, do when you look at *St. Jerome* today. Appropriately enough, we claim that one of the best ways to celebrate the anniversary of this famous etching is to study a bit of perspective geometry.

Figure 2 gives a typical setup for understanding a perspective picture. The idea is that an artist draws a picture by projecting a three-dimensional world onto a two-dimensional canvas so that the world and the canvas appear to line up with each other from where the artist stands. It's obvious that if the artist moves, the world and the canvas will no longer appear to line up with each other—and it's this seemingly obvious fact that plays the most important role in our analysis.
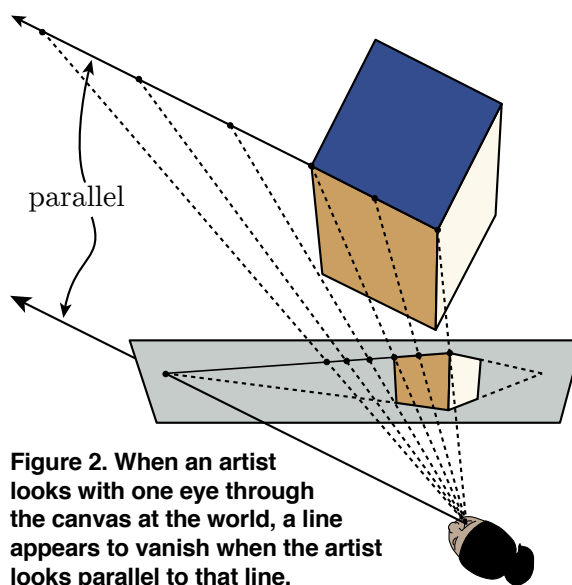

parallel

Figure 2. When an artist looks with one eye through the canvas at the world, a line appears to vanish when the artist looks parallel to that line.
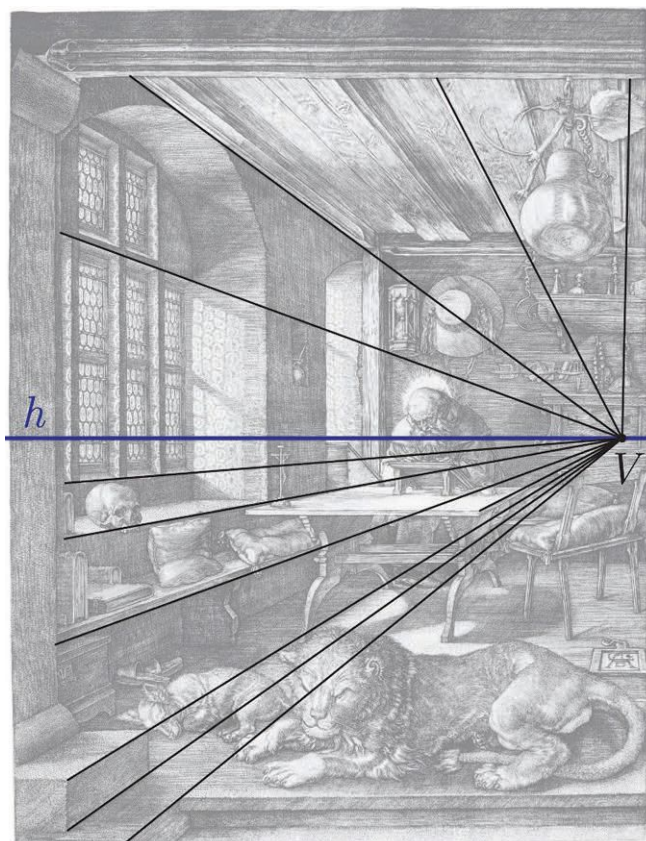
Figure 3. The vanishing point **V** for lines that are perpendicular to the picture plane. The height of vanishing point gives us the horizon line **h**.

Figure 4. If we stand in front of **C**, the table does not appear to be square—otherwise, how could we see the side **AB**?

A well-known consequence of perspective projections is that lines that are parallel in the real world, but not parallel to the picture plane, have images that converge to a point artists call a "vanishing point." Figure 2 shows that if the artist is standing in the right place (that is, the place from which the 3D world and its 2D image line up with each other), then looking at a vanishing point means looking parallel to the lines that gave us the vanishing point in the first place.

This vanishing point is one of our first clues to seeing why Ivins and Dürer disagreed. Figure 3 shows the principal vanishing point *V* for *St. Jerome*, which occurs just to the right and down from the letter "D" that Dürer etched into the saint's cabinet. In particular, *V* is the vanishing point for the lines in St. Jerome's study that are perpendicular to the picture plane. So, by our "seemingly obvious" fact above, this means that the best way to view *St. Jerome* is **not** to hold the picture directly in front of us (as we normally do), but rather to shift it to the left, so our eye is directly in front of the point *V*.

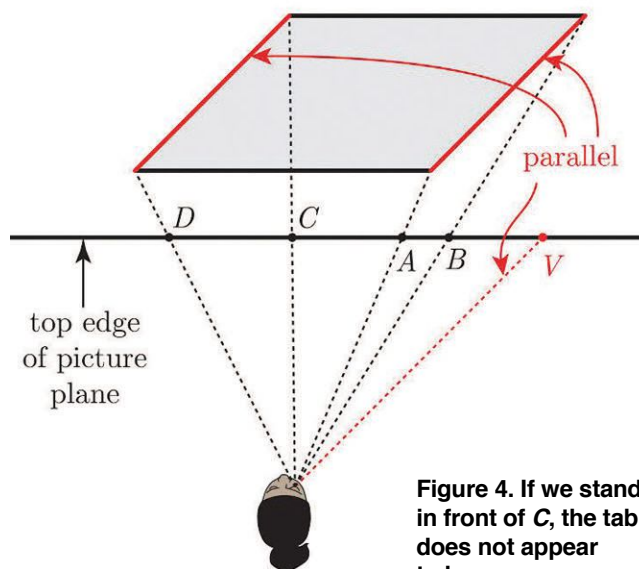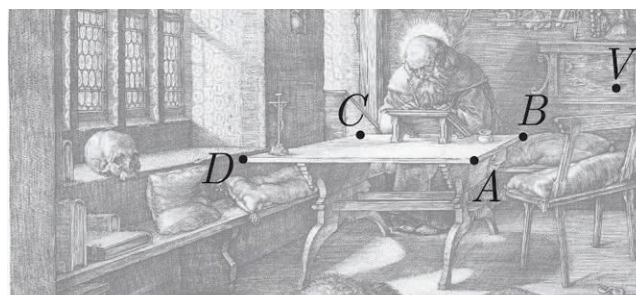Ivins seemed to be aware that there's something not quite right about holding the picture straight ahead. What happens if we make the same mistake? Suppose we look from the front of the table—say, out from the left-rear corner (point *C* in figure 4). But we can still see the right edge of the table (side *AB*). This must mean the table is not a rectangle after all. In fact, as figure 4 shows, the table looks like a parallelogram or trapezoid—just as Ivins said. The oddness that Ivins saw in the table wasn't because *Dürer* was in the wrong, but because Ivins was in the wrong, literally: he was looking from the wrong place!

So where, exactly, should we we hold the engraving to make it appear most correct? We know our eye should be in front of the point *V*, but how far away? A standard perspective solution (as, for example, in [4]) begins with drawing the image of the diagonal line across the top of the table, as in figure 5. If the tabletop is a square, then this line creates a 45° angle with the picture plane (so the line is not parallel to the picture plane, and therefore it has a vanishing point); if the table top is level with the floor, then the image of the diagonal line vanishes on the horizon, at the point at *Z*.

How does this diagonal line help us? We can use the vanishing point *Z* together with the "seemingly obvious" lesson of figure 2: When we look from the right location, our line of sight to the point *V* should be perpendicular
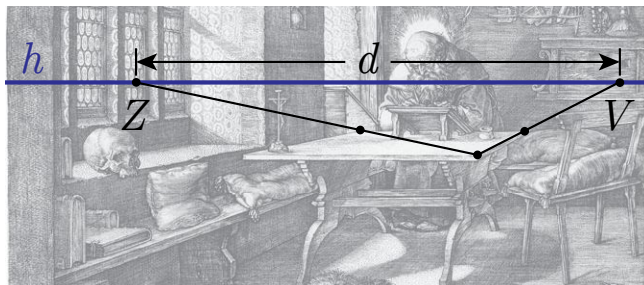
Figure 5. The diagonal across the top of the table has a vanishing point *Z* on the horizon.
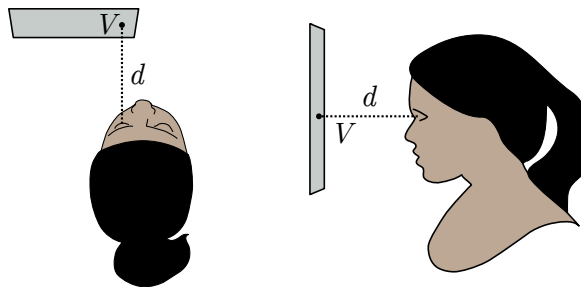

Figure 6. To look at *St. Jerome in His Study* so that it appears most three-dimensional, view it with one eye in front of the point *V*, at a distance *d* from the picture.

to the picture plane, and our line of sight to the point Z should make a 45° angle with the picture plane. That is, just as in figure 6, we should look at *St. Jerome* from the right side, holding the picture close to one eye, at a distance d, which is approximately three-fourths the width of the picture. (Figure 1 is roughly the same size as the original engraving; it was 7.4 inches by 9.7 inches with viewing a distance of $d = 5.6$ inches.)

To return to the Ivins/Dürer controversy, by setting the viewing location unnaturally close to a point on the right edge of the canvas, Dürer certainly ensured that most people would fail to see *St. Jerome in His Study* from the "correct" location. Although there are scholars who have written correctly and carefully about this piece (Martin Kemp [5, pp. 60–61] does a particularly nice job), Ivins was only one of many people who missed out because of Dürer's choice of viewpoint.

For example Patrick Maynard quotes Ivins's paragraph above in his own discussion of *St. Jerome* [6, p. 181]. Maynard acknowledges that the difficulty is attributable in part to "the short distance and eccentric CVP [central vanishing point]." But even Maynard does not seem to relate "the short distance" to the person looking at the picture; he seems to describe the

Please see Dürer, page 25.

Dürer's Homage to Mathematics

Dürer's 1514 *Melencolia I* might very well be his most enduring homage to mathematics. Mathematicians have returned the compliment by writing and publishing numerous articles about this elegant, multifaceted piece. Some of the objects worth noticing in this engraving include

- a 4×4 magic square featuring the date 1514, in which rows, columns, diagonals, center squares, and corners sum to 34;
- a geometer's compass;
- measuring tools (such as a scale and an hourglass); and
- a geometric shape now known as Dürer's solid (the exact shape of this polyhedron has been a subject of many mathematical investigations).
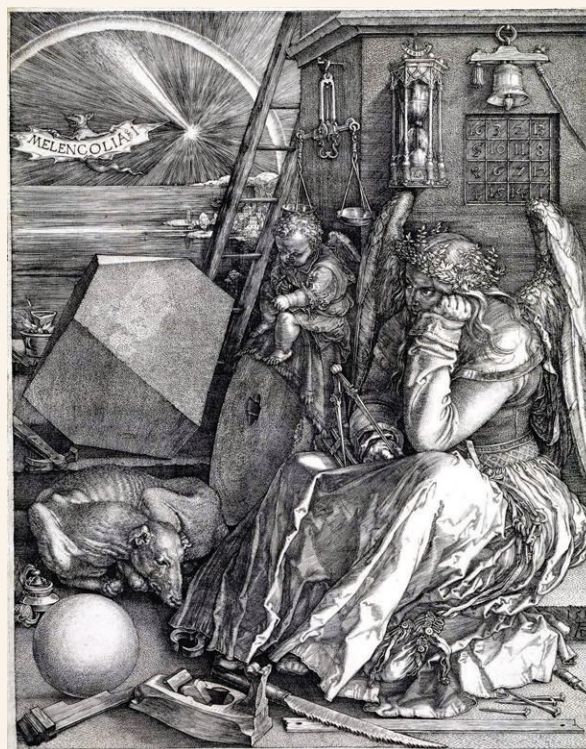

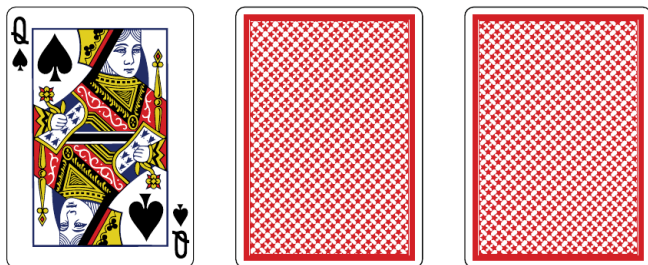**Figure 7. *Melencolia I,* Albrecht Dürer, 1514.**

Figure 5. What is the target card?

Further Reading

[1] Michael Kleber, "The best card trick," *Mathematical Intelligencer* 24, no. 1 (Winter 2002): 9–11.

[2] W. Wallace Lee, *Math Miracles*, 1976 ed., Calgary: Micky Hades International, 1951.

[3] Colm Mulcahy, "Fitch Cheney's five card trick," *Math Horizons* 10 (February 2003): 11–13.

[4] Colm Mulcahy, "Fitch four glory," *Card Colm,* February 2005, *www.maa.org/community/maa-columns/past-columns-card-colm/fitch-four-glory.*

[5] Colm Mulcahy, "Mathematical card tricks," *What's New in Mathematics,* October 2000, *www.ams.org/featurecolumn/archive/mulcahy1.html.*

[6] Shai Simonson and Tara Holm, "Using a card trick to teach discrete mathematics," *PRIMUS* XIII, no. 3 (September 2003): 248–269.

Colm Mulcahy is professor of mathematics at Spelman College, Atlanta, Georgia, and author of Mathematical Card Magic: Fifty-Two New Effects *(A K Peters/CRC, 2013) (see review, page 28). He just completed 10 years of writing the bimonthly* Card Colm *column on mathematical card tricks,* cardcolm-maa.blogspot.com.

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Playing cards © Chris Aguilar

Solutions

2. The card is the 3C.

1. The last card is the target; it is the 2S.

Dürer from page 11.

distortion as a fact particular to this picture, making the picture itself wrong, no matter what.

On the other hand, a deeper understanding of geometry can help us to put ourselves, as viewers, "in the right." That is, we can correct the mistakes that other observers like Ivins and Maynard have made; we can see the effect that the master geometer Albrecht Dürer intended. If you view *St. Jerome in His Study* as indicated in figure 6, you'll see that the engraving takes on an amazing realism and depth. The gourd in the picture seems to hover over your head; you feel you could stick your hand in the space under the table; the bench off to the left invites you to come sit down and fluff up the pillows.

And to see this masterpiece come alive, to move into a space that was created centuries ago—to our mind, that's the perfect way to pay homage to a great mathematician and artist on this 500-year anniversary! ■

References

[1] A. Dürer, *Unterweysung der Messung mit dem Zirckel und Richtscheyt (Instruction in Measurement with Compass and Ruler)* (Nuremberg, Germany: Koberger, 1525).

[2] ———, *Die Proportionslehre (The Theory of Proportion)* (Nuremberg, 1528).

[3] W. Ivins, *On the Rationalization of Sight: de artificiali Perspectiva* (New York: Da Capo Press, 1973).

[4] M. Frantz and A. Crannell, *Viewpoints: Mathematical Perspective and Fractal Geometry in Art* (Princeton: Princeton University Press, 2011).

[5] M. Kemp, *The Science of Art* (New Haven: Yale University Press, 1990).

[6] P. Maynard, *Drawing Distinctions* (New York: Cornell University Press, 2005).

Annalisa Crannell is a professor of mathematics at Franklin & Marshall College who is active in developing materials on mathematics and art. She especially enjoys talking to nonmathematicians who haven't (yet) learned where the most beautiful aspects of the subject lie.

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Marc Frantz majored in painting at the Herron School of Art. He had a 13-year career as a painter and picture framer and subsequently received his M.S. in mathematics from Purdue University. He is coauthor with Annalisa Crannell of the undergraduate text Viewpoints: Mathematical Perspective and Fractal Geometry in Art.

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Fumiko Futamura is an associate professor of mathematics at Southwestern University and also an artist with a focus in oil painting and soft sculpture. She is collaborating with Frantz and Crannell in writing a textbook on projective geometry that focuses on introducing and illuminating concepts through perspective drawing and photography.

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