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Jennifer J. Quinn

“Ladies and gentleman, welcome to Erdős Stadium where combinatorialists from around the world do battle, using their skills to create truly artistic mathematical identities. The winner will gain the people’s ovation and fame forever.”

Massive applause greeted the Chairman’s statement, then he continued. “Representing counting is Erdős Stadium’s resident enumerative expert. She is known to use all her fingers and all her toes. Her favorite technique is to ask a question and answer it in two different ways. You know her. You love her. I present to you . . . the Countess.”

A regal woman in a flowing blue gown glided onstage. She gave a slight smile and gently inclined her head to the audience.

He continued, “If memory serves me, our challenger is known to attack in pairs. He shows no fear of negative signs. His favorite technique of ‘Description—Involution—Exception’ takes his challenger’s strengths and makes them his own. Representing the mathematics of matching, I give you . . . Sir Match-A-Lot.”

An enthusiastic man danced his way down the center aisle playing to the audience. He waved and blew kisses until signaled to a stop.

A black-and-white-shirted official took over the microphone. “The contest will proceed in three timed rounds,” she explained. “A required mystery parameter will be revealed for each round. You must prepare an identity featuring the required component and your own mathematical technique.

“Tonight’s judges are: Tim Possible, known for using probabilistic arguments; Dr. X, the stadium generatingfunctionologist; and Miss Fin de Vol, a pigeon-hole-principle practitioner. I want a good clean battle. Now shake hands. May the best mathematician prevail.”

The Chairman reclaimed the stage. “I raise my coffee to the Father of Combinatorics, Paul Erdős.” He gestured toward a portrait hung center stage, took a sip, and bellowed, “*Allez compute*!” to signal the start of the competition. Two assistants removed drapery to reveal a chalkboard with the words *binomial coefficients*.

Round 1

While the contestants labored in isolation booths on each side of the stage, the audience listened to the judges’ conversation.

“Zis means zee contestants need to present an identity zat features binomial coefficients, and ze proof must utilize zair specific technique of counting or matching, no?” commented Fin de Vol.

Tim Possible responded, “One hundred percent certain, Miss de Vol. Binomial coefficients are practically a gift. With an obvious combinatorial interpretation of $\binom{n}{k}$ as the number of subsets of size k that can be formed from an n-element set, I’d wager that both the Countess and Sir Match-A-Lot will have plenty to bring forward for judgment.”

A buzzer sounded, and the contestants stepped away from their workspaces, hands held high. The official took control. “An off-stage coin toss determined who presents first. Countess, what is your mathematical offering?”

The Countess stepped forward. “My motto is ‘Keep it simple,’ and I strived to do just that. I begin by asking the question, ‘how many subsets does an n-element set contain?’ The first answer to this question is 2^n by considering each element in turn and understanding its two options are to be *in* or *out* of the subset.

"The second answer directly counts subsets by cardinality. There are $\binom{n}{0}$ empty subsets, $\binom{n}{1}$ subsets with 1 element, and in general $\binom{n}{k}$ subsets with k elements. Once k is larger than n, $\binom{n}{k} = 0$. So the number of subsets is the sum of binomial coefficients $\binom{n}{k}$ as k ranges between 0 and n. Because I have answered the same question in two different ways, the answers are equal. Hence, my identity, featuring a simple sum of binomial coefficients, is

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n."$$

Sir Match-A-Lot, eager for his chance to shine, pushed ahead. "That's nice, Countess, but I chose to add drama with some negative signs. Sure, I could have presented $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0$. But that was too easy. Instead, I *generalized.* Let me break it down for you.

"Description: Choose an integer k between 0 and n. Restrict consideration to subsets of an n-element set with k or fewer members. Subsets with an even number of elements contribute $+1$; subsets with an odd number of elements contribute -1. The total weighted contributions from subsets of size less than or equal to k is the alternating sum of binomial coefficients:

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^k\binom{n}{k}.$$

"Involution: Start by picking your favorite of the n elements. Without loss of generality, call it Waldo. Time for the ol' switcheroo. Given any subset, match it by flipping Waldo—meaning that if Waldo is in the set, the matching subset will be the same except boot Waldo out. If Waldo is MIA, the matching subset welcomes him back. This changes the size of a subset by one, sometimes bigger, sometimes smaller, but always pairs an even subset with an odd one. Because $1+(-1)=0$, the matched pairs contribute a big, fat zero. Only the unmatched subsets matter.

"Exceptions: A set of size k with no Waldo cannot switcheroo. There's no elbow room. The subset size is maxed out. Anyone but Waldo could be in such a set, so there are $\binom{n-1}{k}$ exceptions. Each such set will contribute $+1$ or -1 to the total depending on the parity of k. This gives my kickin' identity, the alternating *partial* sum of binomial coefficients:

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^k\binom{n}{k}=(-1)^k\binom{n-1}{k}."$$

The judges conferred before Dr. X kicked off the postgame analysis: "Countess, your presentation was succinct and elegant. Simplicity of thought is admi-

rable, but we were disappointed by the lack of complexity to your identity. You could have been true to yourself while still giving us something more substantial. Why not sums of squares or a convolution?"

"Monsieur Match-A-Lot really took us beyond," Fin de Vol pronounced, "generalizing a partner identity to zee Countess's. He is clever but with language like zat, no one would ever call him elegant."

Tim Possible concluded, "The binomial coefficients were prominently featured in both works, as required. So we judged on style and complexity. While Miss de Vol bristled at Match-A-Lot's nonstandard demeanor, there is no denying his raw enthusiasm. Therefore, by agreement of the judges, Round 1 is awarded to Sir Match-A-Lot."

Round 2

The second-round challenge parameter was revealed to be *Fibonacci numbers.* As the competitors labored, the judges speculated on possible combinatorial interpretations.

Dr. X began, "Assuming $F_0 = 0$, $F_1 = 1$, and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$, F_n is known to count binary $(n-2)$-tuples with no consecutive 0s; subsets of $\{1, 2, \ldots, n-2\}$ with no consecutive integers; or compositions of $n-1$ using one 1s and 2s."

"Mon favorite is tilings of a $1\times(n-1)$-board using squares and dominoes," added Fin de Vol.

Tim Possible chimed in. "There are other fascinating Fibonacci tilings too. For example, tilings of an $(n+1)$-board with tiles of length greater than or equal to 2, or tilings of an n-board where all tiles have odd length. And that is just the tip of the iceberg."

At the conclusion of the interval, Sir Match-A-Lot presented first.

"Because you liked my previous work so much, I decided to Fibonaccize it. My offering is a formulation for an alternating sum of Fibonacci numbers, $F_1 - F_2 + F_3 - \cdots + (-1)^n F_n$. You dig? As you all surely know, F_k counts tilings of $(k-1)$-boards with squares and dominos provided $k \geq 1$.

"Description: Count the square and domino tilings of boards with lengths between 0 and $n-1$. Even-length boards contribute $+1$; odd-length boards contribute -1. The total weighted contributions from the tilings is the alternating sum of consecutive Fibonacci numbers as promised.

"Involution: Switcheroo the last tile. If it's a domino, make it a square and vice versa. Matched tilings add to zero.

"Exceptions: No can do a switcheroo on the empty tiling—of length zero. Zero is even, so it contributes $+1$. No elbow room to switcheroo on length $n-1$ tilings that end in a square. There are F_{n-1}, and they each contribute $(-1)^n$ to the total. The take home? This sweet little ditty:

$$F_1 - F_2 + F_3 - \cdots + (-1)^n F_n = 1 + (-1)^n F_{n-1}.\text{"}$$

He bowed to the audience and winked at the Countess as she made her way to the judging table. She did her best to ignore him.

"Taking your previous admonition to heart," she said to the judges, "I wanted to give you more than a sum of Fibonacci numbers. As my opponent would say, that is *too* simple. Instead, I worked to count the sum of *squares* of Fibonacci numbers by asking 'in how many ways can I tile two different boards—one of length $n-1$ and the other of length n—with squares and dominos?'

"You already know there are F_n ways to tile the first and F_{n+1} ways to tile the second. So the first answer to my question is the product $F_n F_{n+1}$.

"For my second answer, I arrange the boards one under the other, aligned to the left. Then I count based on the rightmost position where breaks in the two tilings coincide vertically." She showed the audience figure 1. "These locations of vertical agreement are termed *faults*. And while I prefer not to seek fault in others, the position of the rightmost (or final) fault is key.

"To its right, tilings are uniquely determined with one board being completed by all dominoes and the other completed with a square followed by dominoes. To its left are two tilings of the same length. If the final fault occurs immediately after cell $k-1$, there are F_k^2 possible tilings.

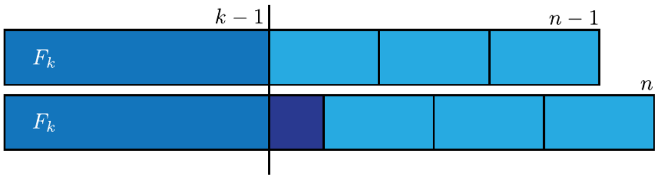


Figure 1. The dominos have a fault after cell *k* − 1.

"Of course, a final fault always exists although it may occur at the very beginning, which I consider *after* the 0th cell. Thus $k-1$, the position of the final fault, ranges between 0 and $n-1$. Summing over all possible values for k gives my second answer and leads to the identity

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.\text{"}$$

Satisfied with her presentation, the Countess took a position beside Sir Match-A-Lot and awaited the judges' response.

"Monsieur Match-A-Lot, zair was a symmetry between your offerings in Round 1 and Round 2. Whereas it served you well originally, it felt redundant this time," commented Fin de Vol. "Madame Countess, you 'eard our criticisms and acted on them. Well done. For complexity of identity, zis round belongs to you in my estimation."

"I don't necessarily agree with your analysis," said Dr. X. "For me, the simplicity of toggling the final tile for the matching absolutely outweighs the fault of faulty tiling pairs."

"In truth, I'm torn," said Tim Possible. "I was impressed by both presentations. So I will settle the question by flipping a coin." He removed a silver dollar from his pocket, threw it into the air, and let it land on the table. "Looks like Round 2 goes to the Countess."

Round 3

Sir Match-A-Lot was not entirely composed for the start of Round 3. He fumed about the arbitrariness of the previous decision. When the mystery parameter was revealed to be the *golden ratio*, he exploded. "Hold the phone! The golden ratio is not an integer quantity. How am I supposed to use $\varphi = (1+\sqrt{5})\,/\,2$ in a matching identity?" He paced Erdős Stadium swearing under his breath for more than half the allotted time.

The Countess, on the other hand, set to work right away, tackling the classic formula credited to Binet but known more than a century earlier to Euler, Bernoulli, and de Moivre. To the audience's delight, both had something to report at the close of the interval. The Countess presented first.

"Judges, my intention is to prove Binet's celebrated formula, $F_n = ((\varphi^n - (-\varphi)^{-n}) / \sqrt{5}$ by asking a question and answering it two different ways. To do this, I use infinite tilings constructed by sequentially and independently placing tiles, selecting squares with probability $1/\varphi$ and dominoes with probability $1/\varphi^2$. Note these are the only two options, as

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1.$$

"I ask, 'what is the probability, q_n, that such an infinite tiling is breakable between cell $n-1$ and cell n?'

"First, I compute the probability directly. Breakability after cell $n-1$ requires that an infinite tiling begins with a square-domino segment of length $n-1$. Each of the F_n possible initial segments occurs with probability $1/\varphi^{n-1}$. So the first answer is $q_n = F_n / \varphi^{n-1}$.

"Next, I compute the probability indirectly by considering the complementary event—when a domino covers cells $n-1$ and n. The tilings *must* be breakable between cells $n-2$ and $n-1$. So the chance of being *unbreakable* after cell $n-1$ is q_{n-1} / φ^2 leading to $q_n = 1 - q_{n-1} / \varphi^2$. Thankfully $q_1 = 1$ and the recurrence unravels into a geometric series

$$q_n = 1 - \frac{1}{\varphi^2} + \frac{1}{\varphi^4} - \frac{1}{\varphi^6} + \cdots + \left(\frac{-1}{\varphi^2}\right)^{n-1}$$

with common ratio $-1/\varphi^2$. Answer two has the closed form

$$q_n = \frac{1-(-1/\varphi^2)^n}{1-(-1/\varphi^2)} = \frac{\varphi}{\sqrt{5}}(1-(-\varphi^2)^{-n}).$$

"The final step is to set answer 1 equal to answer 2 and solve for F_n. Thus I arrive at

$$F_n = \varphi^{n-1} \cdot \frac{\varphi}{\sqrt{5}}(1-(-\varphi^2)^{-n}) = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^{-n}).\text{"}$$

Sir Match-A-Lot, not to be outdone, sauntered center stage and proclaimed, "That was mighty fine indeed, Countess. But your technique was probabilistic and less true to your counting roots. I, on the other hand, will prove the *exact same identity* using matching, and only matching, thereby demonstrating my superiority in this arena. Check. It. Out.

"Description: Tile a $1 \times n$ board with red squares of weight φ, orange squares of weight $\bar{\varphi}$ (aka $-\varphi^{-1}$), and dominoes of weight 1. Be warned, first tiles are special and have weights $\varphi/\sqrt{5}$, $-\bar{\varphi}/\sqrt{5}$, and 0 respectively. Compute the weight of a tiling by multiplying the individual weights. Lay your eyes on this tiling of weight $-\varphi\bar{\varphi}^5/\sqrt{5}$." He presented figure 2.

Figure 2.

"The real deal involves all the weighted tilings of a particular length. If W_n is the sum of the weights for all n-tilings, then W_n is really the Fibonacci number F_n in disguise.

"Don't take my word for it. Do the math. Check initial values. A 1-tiling is either a red square or an orange square. Total weight $W_1 = \varphi/\sqrt{5} + (-\bar{\varphi}/\sqrt{5}) = 1$. A 2-tiling is either two red squares, a red square followed by an orange square, an orange square followed by a red square, two orange squares, or a domino. Total weight

$$W_2 = \frac{\varphi^2}{\sqrt{5}} + \frac{\varphi\bar{\varphi}}{\sqrt{5}} - \frac{\varphi\bar{\varphi}}{\sqrt{5}} - \frac{\bar{\varphi}^2}{\sqrt{5}} + 0 = \frac{\varphi^2}{\sqrt{5}} - \frac{\bar{\varphi}^2}{\sqrt{5}} = 1.$$

"For any tiling longer than 2, it ends with an ordinary tile, either a red square, orange square, or domino. Total weight

$$W_n = \varphi W_{n-1} + \bar{\varphi} W_{n-1} + W_{n-2} = W_{n-1} + W_{n-2}.$$

"You see? These weighted tilings match initial conditions and agree on the recurrence relation. So they represent. Fibonacci numbers that is.

"Involution: The switcheroo will *usually* flip a domino (weight 1) for two squares of different colors (weight $\varphi\bar{\varphi} = -1$), or vice versa. Look for whichever comes first, a domino or a pair of consecutive squares of different colors. As long as it does not involve a leading tile of special weight, do the switch—being sure to match the first square with the color of squares that precede it." And he illustrated this by presenting figure 3.

"The weight of the matched tilings adds to zero, just the way I like it. If a tiling begins with a red square followed by an orange square (contributing a factor of $\varphi\bar{\varphi}/\sqrt{5}$ to the weight), flip the colors so that it begins orange followed by red (contributing a factor of $-\varphi\bar{\varphi}/\sqrt{5}$ to the weight). Again, the weight of the matched tilings adds to zero. If the tiling begins with a domino, its weight is zero by definition, so it will not matter one iota.

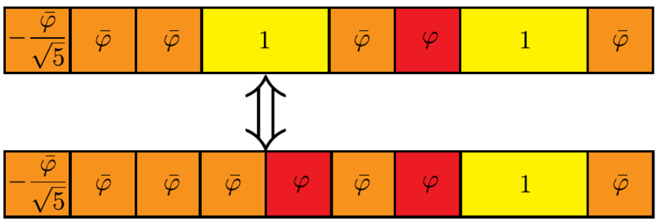
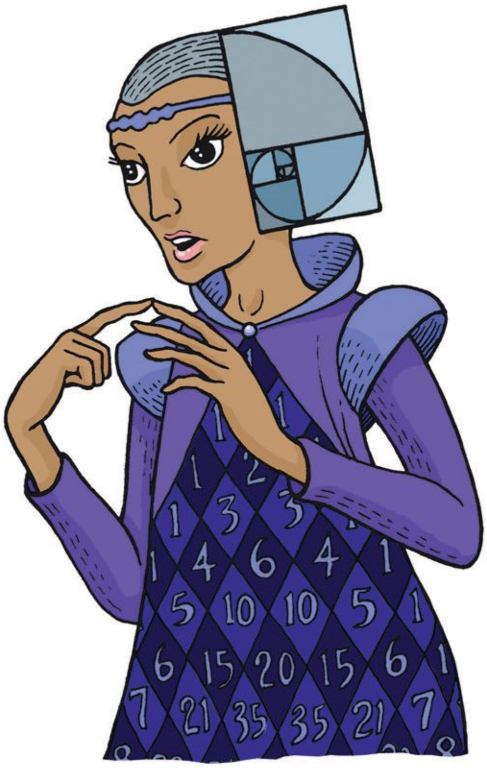


Figure 3.



"Exceptions: Monochromatic squares are the only unmatched, nonzero weight tilings. There are two—all red squares and all orange squares. So,

$$W_n = (F_n =)\ \varphi^n \,/\, \sqrt{5} + (-\overline{\varphi}^n \,/\, \sqrt{5})$$

through the beauty of matchings. I rest my case."

The Chairman led the applause. "Well done. Clearly this is a battle for the history books. Given the current score, the winner of the third and final round will be crowned Epic Math Battles champion. What do our judges have to say?"

Tim Possible was enraptured. "Countess, I could not imagine a more beautiful proof of this classic identity."

Dr. X addressed the Countess. "The balance of big picture and nitty-gritty was reasonable, but I question your choice to gloss over algebraic details. I'm taking some of your leaps on faith—particularly the simplification of the geometric series. And coming from me, that's saying a lot."

Fin de Vol addressed Sir Match-A-Lot, "I was less zan impressed by your early theatrics. Sportsmanship is essential in any competition. Ze weighted tilings seemed an overly complex combinatorial interpretation, but they swayed me when everything fell away except for ze two all-squares tilings and zair weights gave Binet's formula with no further simplification. Both presentations were truly magnifique."

Tim Possible continued, "Rather than leaving the decision to chance, we believe it best to let the audience decide. Who won? The Countess or Sir Match-A-Lot? Counting or matching?"

Cast your vote now at *maa.org/mathhorizons/supplemental.htm*. Voting closes on March 21. The winner will be announced on the *Math Horizons* Facebook and Twitter accounts. ■

Further Reading

A. T. Benjamin and J. J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof* (Washington, DC: Mathematical Association of America, 2003).

A. T. Benjamin and J. J. Quinn, "An Alternate Approach to Alternating Sums: A Method to DIE For," *College Mathematics Journal* 39, no. 3, (2008): 191–201.

A. T. Benjamin, H. Derks, and J. J. Quinn, "The Combinatorialization of Linear Recurrences," *Electronic Journal of Combinatorics* 18, no. 2, (2011): 12.

Jennifer Quinn is a professor of mathematics at the University of Washington Tacoma and past–second vice president of the MAA. She coedited Math Horizons *2004–2008 with Art Benjamin. As a combinatorial scholar, Jenny thinks that simplicity, elegance, and transparency should be driving principles—along with a dash of fun.*

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