The Early History of Block Designs

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In this article I outline the origins of design theory, with particular reference to the history of Steiner triple systems, the work of Kirkman, and early work on the 'schoolgirls problem'. I am grateful to Norman Biggs and Terry Griggs for allowing me access to their unpublished work in this area.

1. A prize question

In the *Lady's and Gentleman's Diary* for 1844 the editor, the Revd. Wesley Woolhouse posed the following problem:

> *Prize Question 1733: Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other.*

The *Diary* was

> *designed principally for the amusement and instruction of students in mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit,*

and readers were invited to send in solutions of the question posed. Just two solutions were received: one misunderstood the problem, and the other, by Septimus Tebay of the Gas Works, Preston, gave the answer ${}^{n}C_{q}$ / ${}^{p}C_{q}$, which is correct when all the q-combinations appear in some p-combination.

This was not the interpretation that Woolhouse had intended, and he posed the question again in 1846, in the specific case $p = 3$, $q = 2$:

> *Question 1760: How many triads can be made out of n symbols, so that no pair of symbols shall be comprised more than once amongst them?*

An example of such a system, where each pair of symbols appears *exactly once* is the following, with $n = 7$; the seven triads, or triples, are presented vertically:

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Such a design is now known as a *triple system* – or a *Steiner triple system*, for reasons that will become apparent; a Steiner triple system with n points will be denoted by $S(n)$. The above design is also a finite projective plane of order 2 – an arrangement of 7 points and 7 lines with exactly three points on each line and three lines through each point; more generally, a finite projective plane of order k is an arrangement of $k^2 + k + 1$ points and $k^2 + k + 1$ lines with exactly $k + 1$ points on each line and $k + 1$ lines through each point.

2. Steiner triple systems

Another example of a triple system, with $n = 9$ and with 12 triples, is as follows:

1	1	1	1	2	2	2	3	3	3	4	7
2	4	5	6	4	5	6	4	5	6	5	8
3	7	9	8	9	8	7	8	7	9	6	9

Note that the frequency f of each number is $(n - 1)/2$, because the other $n - 1$ numbers appear in pairs with it. Also, when there are t triples, the total number of entries in the system is $3t = nf$, so $t = n(n - 1)/6$. Since this is an integer, $n \equiv 1$ or 3 (mod 6). Initially it was not known whether a triple system can be constructed for each such value of n, but this is indeed the case – there is essentially one system for $n = 7$ or 9, two systems for $n = 13$, eighty for $n = 15$, and millions for all higher values ($n = 19, 21, 25, \ldots$).

For future reference, we call a triple system *resolvable* if its triples can be rearranged into subsystems, each containing all n numbers. For example, the above system for $n = 9$ is resolvable, as the following rearrangement shows:

1	4	7	|	1	2	3	|	1	2	3	|	1	2	3
2	5	8	|	4	5	6	|	6	4	5	|	5	6	4
3	6	9	|	7	8	9	|	8	9	7	|	9	7	8

3. The work of Plücker

In the 1830s, Julius Plücker was studying plane cubic curves of order 3. In his 1835 book [1] he observed that a general plane cubic curve has 9 points of inflection that lie in threes on 12 lines; moreover, given any two points of the system, exactly one of the lines passes through them both. This triple system is now known as the *9-point affine plane*, and the lines correspond to the twelve triples are as given above. Plücker presented the system explicitly, and in a footnote he observed that:

> *If a system S(n) of n points can be arranged in triples, so that any two points lie in just one triple, then n ≡ 3 (mod 6) ...;*

in a later book [2] he corrected his mistake, adding $n \equiv 1 \pmod 6$.

It is a matter of conjecture as to how Wesley Woolhouse came to consider triple systems, but it is possible that James Joseph Sylvester, who had a life-long interest in combinatorial systems and wrote on 'combinatorial aggregation' in 1844 [3], knew of Plücker's work and mentioned it to Woolhouse.

Many years later, in 1892, the ideas of Plücker were developed in a celebrated geometrical paper of Gino Fano [4]. In this work, he discussed the 7-point projective plane, later known as the *Fano plane*, and described projective planes with 13 and higher numbers of points.

4. Kirkman's 1847 paper

The Revd. Thomas Pennington Kirkman was a keen mathematician who had studied in Dublin, and was Rector of the small parish of Croft-with-Southworth in Lancashire. His parochial duties took up little of his time, and he concentrated much effort on his mathematical researches, especially on algebraic and combinatorial topics.

On 15 December 1846, Kirkman read a paper to the Literary and Philosophical Society of Manchester, entitled *On a problem in combinations*. In this pioneering paper, published the following year in the *Cambridge and Dublin Mathematical Journal* [5], Kirkman showed how to construct a Steiner triple system $S(n)$ for each possible integer n. Specifically, he introduced a supplementary system $D(2m)$ which is an arrangement into $2m - 1$ columns of the ${}^{2m}C_2$ pairs of $2m$ symbols; these systems had earlier been considered by Sylvester in his 1844 paper [3]. The system D_8 is as follows: it is sometimes called a *Room square*, and can also be regarded as an edge-colouring of the complete graph K_8, with the columns listing those sets of four edges that are assigned the same colour.

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
kl	*il*	*ik*	*in*	*im*	*ip*	*io*
mn	*mo*	*mp*	*ko*	*kp*	*mk*	*nk*
op	*np*	*no*	*lp*	*lo*	*nl*	*ml*

Kirkman then presented two constructions, in which he used these systems to extend smaller Steiner systems to larger ones:

- from a Steiner triple system $S(n)$ and a supplementary system $D(n+1)$, he constructed a Steiner triple system $S(2n+1)$; for example, from the Steiner system $S(7)$ and the above design $D(8)$, he constructed a Steiner system $S(15)$;
- from a Steiner triple system $S(2k + 1)$, he removed two symbols to obtain a 'reduced system' $S^*(2k - 1)$: then, from any reduced system $S^*(n + 1)$ he constructed a Steiner system $S(2n + 1)$ and a reduced design $S^*(2n - 1)$; for example, from the Steiner system $S(7)$, he successively obtained $S^*(5)$ and $S(9)$.

By combining these constructions, Kirkman constructed a Steiner triple system with n points, for each $n \equiv 1$ or 3 (mod 6).

In later papers he developed these ideas. For example, in [6] he showed how to construct projective planes of order p (for any prime p) and orders 4 and 8, and in the very obscure *Transactions of the Historical Society of Lancashire and Cheshire* [7] he invented cyclic difference sets and used them to construct projective planes of orders 2, 3, 4 and 5.

5. The contribution of Steiner

In 1853, the celebrated geometer Jakob Steiner wrote a short note [8] on triple systems, a topic that he probably encountered through his study of Plücker's work. In this note, Steiner correctly observed that triple systems with n points can exist only when $n = 1$ or 3 (mod 6), and asked for which numbers n such triple systems can be constructed, completely unaware that Kirkman had completely solved this problem six years earlier; this lack of awareness probably arises from the fact that the *Cambridge and Dublin Mathematical Journal*, though well known in Britain, was little known on the Continent. The situation was further complicated when M. Reiss [9] solved Steiner's problem using methods very similar to those of Kirkman, causing the latter to complain sarcastically in [10]:

> *... how did the Cambridge and Dublin Mathematical Journal ... contrive to steal so much from a later paper in Crelle's Journal, Vol. LVI., p. 326, on exactly the same problem in combinations?*

The term 'Steiner triple system' was later coined by Ernst Witt. Thus, not only did Kirkman fail to gain the credit for 'Hamiltonian' graphs which should rightly have been named after him (see [11, Ch. 2]), he also 'missed out' on receiving the credit for his fundamental work on triple systems. Instead, he is remembered mainly for a recreational puzzle that arose out of his work on triple systems.

6. Kirkman's schoolgirls' problem

While preparing his 1847 paper, Kirkman noticed that the 35 triples of his system $S(15)$ can be split into subsystems, each containing all 15 points – that is, it is resolvable. In the *Lady's and Gentleman's Diary* for 1850 [12], intermingled with challenges on the sons of Noah and the origins of April Fool's Day, he proposed a recreational form of this observation, now known as 'Kirkman's schoolgirls problem':

> *Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.*

A solution of this problem is:

Monday:	*1-2-3;*	*4-5-6;*	*7-8-9;*	*10-11-12;*	*13-14-15*
Tuesday:	*1-4-7;*	*2-5-8;*	*3-12-15;*	*6-10-14;*	*9-11-13*
Wednesday:	*1-10-13;*	*2-11-14;*	*3-6-9;*	*4-8-12;*	*5-7-15*
Thursday:	*1-5-11;*	*2-6-12;*	*3-7-13;*	*4-9-14;*	*8-10-15*
Friday:	*1-8-14;*	*2-9-15;*	*3-4-10;*	*6-7-11;*	*5-12-13*
Saturday:	*1-6-15;*	*2-4-13;*	*3-8-11;*	*5-9-10;*	*7-12-14*
Sunday:	*1-9-12;*	*2-7-10;*	*3-5-14;*	*6-8-13;*	*4-11-15*

Kirkman's solution appeared in the *Diary* for 1851. He incorrectly claimed that it was 'the symmetrical and the only possible solution', as in the meantime Arthur Cayley [13] had produced a different solution, still with a lot of symmetry. Also, in the 1851 *Diary*, a single solution was given by Mr Bills of Newark-on-Trent, Mr Jones of Chester, Mr Wainman of Leeds and Mr Levy of Hungerford – it is not known how they all supposedly arrived at the same solution.

Kirkman presented variations of the schoolgirls problem, solving the corresponding problem for nine young ladies, believing it to be impossible for 21 young ladies, and asserting the following results [14]:

> *Sixteen young ladies can all walk out four abreast, till every three have once walked abreast; so can thirty-two, and so can sixty-four young ladies; so can 4^n young ladies.*

In 1852, Spottiswoode [15] extended the problem to $(2^{2p} - 1)/3$ young ladies – that is, to numbers in the sequence 15, 63, 225, 1023, ...

7. Further solutions

In 1852 a new type of solution was produced by the Revd. Robert Anstice [16]. Unlike the previous somewhat *ad hoc* attempts, Anstice sought systems that involved some structure, and he produced a cyclic solution. Here the fifteen schoolgirls can be written as *0*, *1*, *2*, *3*, *4*, *5*, *6*, $\underline{0}$, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, $\underline{5}$, $\underline{6}$ and ∞, and each successive day's arrangement is obtained by adding 1 to each entry of the previous day, as follows:

Monday:	∞-0-$\underline{0}$;	$\underline{1}$-2-$\underline{3}$;	1-$\underline{4}$-$\underline{5}$;	3-5-6;	4-$\underline{2}$-$\underline{6}$
Tuesday:	∞-1-$\underline{1}$;	$\underline{2}$-3-$\underline{4}$;	2-$\underline{5}$-$\underline{6}$;	4-6-0;	5-$\underline{3}$-$\underline{0}$
Wednesday:	∞-2-$\underline{2}$;	$\underline{3}$-4-$\underline{5}$;	3-$\underline{6}$-$\underline{0}$;	5-0-1;	6-$\underline{4}$-$\underline{1}$
Thursday:	∞-3-$\underline{3}$;	$\underline{4}$-5-$\underline{6}$;	4-$\underline{0}$-$\underline{1}$;	6-1-2;	0-$\underline{5}$-$\underline{2}$
Friday:	∞-4-$\underline{4}$;	$\underline{5}$-6-$\underline{0}$;	5-$\underline{1}$-$\underline{2}$;	0-2-3;	1-$\underline{6}$-$\underline{3}$
Saturday:	∞-5-$\underline{5}$;	$\underline{6}$-0-$\underline{1}$;	6-$\underline{2}$-$\underline{3}$;	1-3-4;	2-$\underline{0}$-$\underline{4}$
Sunday:	∞-6-$\underline{6}$;	$\underline{0}$-1-$\underline{2}$;	0-$\underline{3}$-$\underline{4}$;	2-4-5;	3-$\underline{1}$-$\underline{5}$

Further cyclic solutions were later produced by Benjamin Peirce [17], who showed that there are three types of cyclic solution corresponding to the following arrangements for *Monday*:

Anstice:	∞-0-$\underline{0}$;	$\underline{1}$-2-$\underline{3}$;	1-$\underline{4}$-$\underline{5}$;	3-5-6;	4-$\underline{2}$-$\underline{6}$
Cayley:	∞-0-$\underline{0}$;	$\underline{1}$-$\underline{3}$-4;	2-$\underline{4}$-$\underline{5}$;	3-5-6;	1-$\underline{2}$-$\underline{6}$
Kirkman:	∞-0-$\underline{0}$;	$\underline{1}$-5-$\underline{6}$;	$\underline{3}$-$\underline{4}$-6;	1-2-4;	3-$\underline{2}$-$\underline{5}$

8. The contributions of Sylvester

As we noted earlier, Sylvester had written about combinatorial systems. In a paper of 1861 [18], he tried to claim priority for the idea of the schoolgirls' problem:

> *... long before the publication of my unfinished paper in the magazine, I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all the duads to base 15, which has since become so well-known, and fluttered so many a gentle bosom, under the title of the fifteen schoolgirls' problem; and it is not improbable that the question, under its existing form, may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge long years before its first appearance, which I believe was in the* Ladies' Diary *for some year which my memory is unable to furnish.*

Kirkman quickly dismissed his claims:

My distinguished friend Professor Sylvester …volunteers en passant *an hypothesis as to the possible origin of this noted puzzle under its existing form. No man can doubt, after reading his words, that he was in possession of the property in question of the number 15 when he was an Undergraduate at Cambridge. But the difficulty of tracing the origin of the puzzle, from my own brains to the fountain named at that university, is considerably enhanced by the fact that, when I proposed the question in 1849, I had never had the pleasure of seeing either Cambridge or Professor Sylvester. My own account of the origin of the problem may be seen at p. 260, vol. v., of the Cambridge and Dublin Mathematical Journal, 1850. No other account of it has, so far as I know, been published in print except this guess of Prof. Sylvester's in 1861.*

Sylvester also proposed that:

It were much to be desired that some one would endeavour to collect and collate the various solutions that have been given of the noted 15-school-girl problem by Messrs Kirkman …, Moses Ansted [presumably Robert Anstice] … , by Messrs Cayley and Spottiswoode …, and Professor Pierce [Peirce], the latest and probably the best …

This was eventually done in a paper of F. N. Cole [19] in 1922. It turns out that there are essentially seven different solutions of the schoolgirls problem. A full bibliography of papers relating to the schoolgirls problem appears in Eckenstein [20].

However, Sylvester did make one lasting contribution to the subject. As reported by Cayley [12] in 1850, Sylvester noticed that there are ${}^{15}C_3 = 455 = 13 \times 35$ possible triples of schoolgirls, and asked whether it is possible to produce thirteen disjoint solutions to the schoolgirls problem, over thirteen weeks, so that each of the 455 triples occurs just once in the quarter-year. Kirkman claimed (wrongly) to have a solution, and the problem remained unsolved for over a hundred years, until it was settled in the affirmative by Denniston [21].

Meanwhile, the general problem for $6n + 3$ schoolgirls remained unsolved until, around the same time, it was settled by Ray-Chaudhuri and Wilson [22], and independently about eight years earlier by Lu Xia Xi, a schoolteacher from Inner Mongolia. The corresponding Sylvester problem for disjoint solutions for $6n + 3$ schoolgirls remains unsolved to this day.

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