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$p < q$, then for every $n \geq 1$, both $p, q \in B^n_{f(p,n)}$. Consider a rational number r_m such that $p < r_m < q$. Since $p \in B^{m+1}_{f(p,m+1)} \subset (0, r_m)$, it follows that $q \notin B^{m+1}_{f(p,m+1)}$, which is inconsistent with $q \in B^n_{f(p,n)}$ for every $n \geq 1$.

Evidently, $F(p) = \{p\}$ for every irrational number $p \in I$. Since there is an uncountable number of irrational numbers in I and distinct irrational numbers yield distinct paths in the tree, there must be an uncountable number of paths in the tree.

Remarks.

1. If, instead of removing the sequence $\{r_n\}_{n=1}^{\infty}$ of rationals in I, the tree were modeled by removing the sequence of numbers $\{2^{-n}\}_{n=1}^{\infty}$ in I, then there would exist just a countable infinity of paths; namely, for each integer $k \geq 1$ there is a path whose intersection is the interval $(2^{-k}, 2^{-k+1})$, and there is a path whose intersection is the empty set.

2. Similarly, if the terms of the removed sequence—say a_n—are strictly decreasing in magnitude as n increases, then there are just a countable infinity of paths; there are those with intersection (a_k, a_{k-1}) for every integer $k \geq 1$ (with a_0 defined to be 1), and a path whose intersection is the interval $(0, \lim a_k)$, which is possibly degenerate.

3. More generally, suppose the sequence of numbers that are removed from I is eventually decreasing or eventually increasing (i.e., decreasing or increasing on a tail of the sequence or for all terms past some fixed term of the sequence) relative to the usual order on the real line. Then there are only a countable number of paths.

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Tabular Integration by Parts

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Only a few contemporary calculus textbooks provide even a cursory presentation of tabular integration by parts [see for example, G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, Addison-Wesley, Reading, MA, 1988]. This is unfortunate because tabular integration by parts is not only a valuable tool for finding integrals but can also be applied to more advanced topics including the derivations of some important theorems in analysis.

The technique of tabular integration allows one to perform successive integrations by parts on integrals of the form

$$\int F(t)G(t)\,dt \tag{1}$$

without becoming bogged down in tedious algebraic details [V. N. Murty, Integration by parts, *Two-Year College Mathematics Journal* 11 (1980) 90–94]. There are several ways to illustrate this method, one of which is diagrammed in Table 1. (We assume throughout that F and G are "smooth" enough to allow repeated differentation and integration, respectively.)

Table 1

Column #1	Column #2
$+F$	G
$-F^{(1)}$	$G^{(-1)}$
$+F^{(2)}$	$G^{(-2)}$
$-F^{(3)}$	$G^{(-3)}$
$\vdots$	$\vdots$
$(-1)^n F^{(n)}$	$G^{(-n)}$
$(-1)^{n+1} F^{(n+1)}$	$G^{(-n-1)}$

In column #1 list $F(t)$ and its successive derivatives. To each of the entries in this column, alternately adjoin plus and minus signs. In column #2 list $G(t)$ and its successive antiderivatives. (The notation $G^{(-n)}$ denotes the nth antiderivative of G. Do not include an additive constant of integration when finding each antiderivative.) Form successive terms by multiplying each entry in column #1 by the entry in column #2 that lies just *below* it. The integral (1) is equal to the sum of these terms. If $F(x)$ is a polynomial, then there will be only a finite number of terms to sum. Otherwise the process may be truncated at any level by forming a remainder term defined as the integral of the product of the entry in column #1 and the entry in column #2 that lies directly *across* from it. Symbolically,

$$\begin{aligned}\int F(t)G(t)\,dt &= FG^{(-1)} - F^{(1)}G^{(-2)} + F^{(2)}G^{(-3)} - \cdots \\ &\quad + (-1)^n F^{(n)}G^{(-n-1)} + (-1)^{n+1}\int F^{(n+1)}(t)G^{(-n-1)}(t)\,dt \\ &= \sum_{k=0}^{n} (-1)^k F^{(k)}G^{(-k-1)} + (-1)^{n+1}\int F^{(n+1)}(t)G^{(-n-1)}(t)\,dt. \quad (2)\end{aligned}$$

A proof follows from continued application of the formula for integration by parts [K. W. Folley, Integration by parts, *American Mathematical Monthly* 54 (1947) 542–543].

The technique of tabular integration by parts makes an appearance in the hit motion picture *Stand and Deliver* in which mathematics instructor Jaime Escalante of James A. Garfield High School in Los Angeles (portrayed by actor Edward James Olmos) works the following example.

Example. $\int x^2 \sin x\,dx$

column #1	column #2
$+x^2$	$\sin x$
$-2x$	$-\cos x$
$+2$	$-\sin x$
0	$\cos x$

Answer: $-x^2 \cos x + 2x \sin x + 2\cos x + C$

The following are some areas where this elegant technique of integration can be applied.

Miscellaneous Indefinite Integrals. Most calculus textbooks would treat integrals of the form

$$\int \frac{P(t)}{(at+b)^r}\,dt,$$

where $P(t)$ is a polynomial, by time-consuming algebraic substitutions or tedious partial fraction decompositions. However, tabular integration by parts works particularly well on such integrals (especially if r is not a positive integer).

Example.

$$\int \frac{12t^2+36}{\sqrt[5]{3t+2}}\,dt$$

column #1	column #2
$+\ 12t^2+36$	$(3t+2)^{-1/5}$
$-\ \ 24t$	$\frac{5}{12}(3t+2)^{4/5}$
$+\ \ 24$	$\frac{25}{324}(3t+2)^{9/5}$
0	$\frac{125}{13608}(3t+2)^{14/5}$

Answer: $(5t^2+15)(3t+2)^{4/5} - \dfrac{50t}{27}(3t+2)^{9/5} + \dfrac{125}{567}(3t+2)^{14/5} + C$

Laplace Transforms. Computations involving the Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{3}$$

often require several integrations by parts because of the form of the integrand in (3). Tabular integration by parts streamlines these integrations and also makes proofs of operational properties more elegant and accessible. (Many introductory differential equations textbooks omit formal proofs of these properties because of the lengthy detail involved in their derivations.) The following example uses this technique to establish the fundamental formula for the Laplace transform of the nth derivative of a function.

Theorem. *Let n be a positive integer and suppose that $f(t)$ is a function such that $f^{(n)}(t)$ is piecewise continuous on the interval $t \geq 0$. Furthermore suppose that there exist constants A, b, and M such that*

$$|f^{(k)}(t)| \leq Ae^{bt} \text{ if } t \geq M$$

for all $k = 0, 1, 2, \ldots, n-1$. Then if $s > b$,

$$L\{f^{(n)}(t)\} = -f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-1}f(0) + s^n L\{f(t)\}. \tag{4}$$

Proof.

column #1	column #2
$+ \quad e^{-st}$	$f^{(n)}(t)$
$- \; -se^{-st}$	$f^{(n-1)}(t)$
$+ \quad s^2e^{-st}$	$f^{(n-2)}(t)$
$\vdots$	$\vdots$
$(-1)^{n-1}(-1)^{n-1}s^{n-1}e^{-st}$	$f^{(1)}(t)$
$(-1)^n(-1)^n s^n e^{-st}$	$f(t)$

$$L\{f^{(n)}(t)\} = \left[e^{-st}f^{(n-1)}(t) + se^{-st}f^{(n-2)}(t) + \cdots + s^{n-1}e^{-st}f(t)\right]_{t=0}^{t=\infty}$$
$$+ \int_0^\infty s^n e^{-st} f(t)\,dt$$
$$= -f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-1}f(0) + s^n L\{f(t)\}.$$

Taylor's Formula. Tabular integration by parts provides a straightforward proof of Taylor's formula with integral remainder term.

Theorem. *Suppose $f(t)$ has $n+1$ continuous derivatives throughout an interval containing a. If x is any number in the interval then*

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$+ \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt. \tag{5}$$

Proof.

	column #1	column #2
$+$	$-f^{(1)}(t)$	-1
$-$	$-f^{(2)}(t)$	$(x-t)$
$+$	$-f^{(3)}(t)$	$-\dfrac{(x-t)^2}{2!}$
	$\vdots$	$\vdots$
$(-1)^{n+1}$	$-f^{(n)}(t)$	$(-1)^n\dfrac{(x-t)^{n-1}}{(n-1)!}$
$(-1)^{n+2}$	$-f^{(n+1)}(t)$	$(-1)^{n+1}\dfrac{(x-t)^n}{n!}$

$$\int_a^x [-f^{(1)}(t)][-1]\,dt$$
$$= \left[-f^{(1)}(t)(x-t) - \frac{f^{(2)}(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n\right]_{t=a}^{t=x}$$
$$+ \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

Equation (5) follows immediately.

Residue Theorem for Meromorphic Functions. Tabular integration by parts also applies to complex line integrals and can be used to prove the following form of the residue theorem.

Theorem. *Suppose $f(z)$ is analytic in $\mathscr{D} = \{z\colon 0 < |z - z_0| < R\}$ and has a pole of order m at z_0. Then if* $0 < r < R$

$$\oint_{|z-z_0|=r} f(z)\, dz = \frac{2\pi i}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]. \tag{6}$$

Proof.

column #1	column #2
$+\ (z-z_0)^m f(z)$	$(z-z_0)^{-m}$
$-\dfrac{d}{dz}(z-z_0)^m f(z)$	$-\dfrac{(z-z_0)^{-m+1}}{m-1}$
$+\dfrac{d^2}{dz^2}(z-z_0)^m f(z)$	$\dfrac{(z-z_0)^{-m+2}}{(m-1)(m-2)}$
$\vdots$	$\vdots$
$(-1)^{m-1}\dfrac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)$ --------→	$(-1)^{m-1}\dfrac{(z-z_0)^{-1}}{(m-1)!}$

Thus,

$$\oint_{|z-z_0|=r} (z-z_0)^m f(z) (z-z_0)^{-m}\, dz$$

$$= \left[-\sum_{k=0}^{m-2} \frac{(m-k-2)!\dfrac{d^k}{dz^k}(z-z_0)^m f(z)}{(m-1)!(z-z_0)^{m-k-1}} \right] \Bigg|_{\oint\, |z-z_0|=r}$$

$$+ \frac{1}{(m-1)!} \oint_{|z-z_0|=r} \frac{\dfrac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)}{z - z_0}\, dz, \tag{7}$$

where the summation term in (7) must be evaluated for the closed circle $|z - z_0| = r$. However, each of the functions in column #1 has a removable singularity at z_0 and is therefore single-valued in $\mathscr{D}$. Furthermore, each of the functions in column #2 is also single-valued in $\mathscr{D}$. Thus, the summation term in (7) must vanish. The integral term on the right side of (7) is evaluated by the Cauchy integral formula and the result (6) follows directly. (The right-hand side of (6) can be recognized as the formula for the residue of $f(z)$ at the pole z_0.) See [Ruel V. Churchill and James W. Brown, *Complex Variables and Applications*, McGraw-Hill, New York, 1984].