A Shortcut in Partial Fractions

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absolute values as

\[ \ln|x + 2|, \]

which, however, does not indicate that, because of the initial condition, \( x + 2 \) must have the same sign as \(-3\). In general, without absolute values,

\[ \int_a^b \frac{1}{x} \, dx = \ln \frac{b}{a}, \quad \text{provided that } ab > 0. \]

References


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The method of partial fractions is the basic technique for preparing rational functions for integration. It is also a useful tool for finding inverse Laplace transforms. This method enables us to write a rational function as a sum of simpler quotients that can be integrated directly or transformed easily by the inverse Laplace operator.

The basic technique to find partial fractions for a rational function is based on the method of undetermined coefficients. However, the computation involved in this method is often tedious. The following is a simple shortcut to expanding certain rational functions in partial fractions. We believe it is worthwhile to include this method in the texts.

**Shortcut.** Let \( p(x) \) be a function and \( a, b \) distinct scalars. Then

\[
\frac{1}{(p(x) + a)(p(x) + b)} = \left( \frac{1}{p(x) + a} - \frac{1}{p(x) + b} \right) \frac{1}{b - a}.
\]

This is a special case of a general algebraic identity, and it is really useful. Let us look at some applications.

**Example 1.**

\[
\frac{x}{(x^2 + 1)(x^2 + 4)} = \frac{x}{4 - 1} \left( \frac{1}{x^2 + 1} - \frac{1}{x^2 + 4} \right) = \frac{1}{3} \left( \frac{x}{x^2 + 1} \right) - \frac{1}{3} \left( \frac{x}{x^2 + 4} \right).
\]
Example 2.

\[
\int \frac{5x - 3}{(x + 1)(x - 3)} \, dx = \int \frac{5(x + 1) - 8}{(x + 1)(x - 3)} \, dx
\]

\[
= \int \left( \frac{5}{x - 3} - \frac{8}{(x + 1)(x - 3)} \right) \, dx
\]

\[
= \int \left( \frac{5}{x - 3} - \frac{8}{4} \left( \frac{1}{x - 3} - \frac{1}{x + 1} \right) \right) \, dx
\]

\[
= \int \left( \frac{3}{x - 3} + \frac{2}{x + 1} \right) \, dx
\]

\[
= 3 \ln|x - 3| + 2 \ln|x + 1| + C.
\]

Example 3.

\[
\int \frac{1}{(x^2 + x + \frac{3}{4})(x^2 + x + \frac{9}{4})} \, dx = \int \left( \frac{1}{x^2 + x + \frac{3}{4}} - \frac{1}{x^2 + x + \frac{9}{4}} \right) \, dx
\]

\[
= \int \left( \frac{1}{(x + \frac{1}{2})^2 + 1} - \frac{1}{(x + \frac{3}{2})^2 + 2} \right) \, dx
\]

\[
= \tan^{-1} \left( x + \frac{1}{2} \right) - \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x + \frac{3}{2}}{\sqrt{2}} \right) + C.
\]

Example 4.

\[
\frac{1}{s^2(s - a)} = \frac{s + a}{s^2(s - a)^2}
\]

\[
= \frac{s + a}{a^2} \left( \frac{1}{s^2 - a^2} - \frac{1}{s^2} \right)
\]

\[
= \frac{1}{a^2} \left( \frac{1}{s - a} \right) - \frac{1}{a^2} \left( \frac{s + a}{s^2} \right)
\]

\[
= \frac{1}{a^2} \left( \frac{1}{s - a} \right) - \frac{1}{a^2} \cdot \frac{1}{s} - \frac{1}{a} \cdot \frac{1}{s^2}.
\]
Example 5. Find the inverse Laplace transform for $1/s(s^2 + 1)(s^2 + 2)$.

Solution. Since
\[
\frac{1}{s(s^2 + 1)(s^2 + 2)} = \frac{1}{s} \left( \frac{1}{s^2 + 1} - \frac{1}{s^2 + 2} \right)
\]
\[
= \frac{s}{s^2(s^2 + 1)} - \frac{s}{s^2(s^2 + 2)}
\]
\[
= \frac{1}{s^2} \left( \frac{s}{s^2 + 1} \right) - \frac{1}{2} \left( \frac{s}{s^2 + 2} \right)
\]
\[
= \frac{1}{2} \cdot \frac{1}{s} - \frac{s}{s^2 + 1} + \frac{1}{2} \left( \frac{s}{s^2 + 2} \right)
\]
we have
\[
\mathcal{L}^{-1} \left( \frac{1}{s(s^2 + 1)(s^2 + 2)} \right) = \frac{1}{2} - \cos t + \frac{1}{2} \cos \sqrt{2} t.
\]

Exercise. Obtain the analogous identity for
\[
\frac{1}{(p(x) + a)(p(x) + b)(p(x) + c)}.
\]