deadly—correspond to the skills required by real applications: reading and understanding problems stated in prose, drawing figures, translating prose into mathematics, setting up multiple integrals, using algebraic or trigonometric identities, and so forth. Furthermore, the case study shows that while calculus problems may seem dull or abstract without documentation by case studies, their applications may nevertheless be quite slick.

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Disks, Shells, and Integrals of Inverse Functions

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Most calculus students (and many calculus textbooks) take it for granted that the shell and disk methods for computing the volume of a solid of revolution must always give the same result. There are really two problems here: to show that the two methods agree, and that the result is the volume of the solid.

Once one has defined volume to be the result of integrating the constant function 1 over solid regions, it is easy to show that the shell method gives the volume, by using cylindrical coordinates. It remains to be shown that the disk method gives the same result. Several elementary proofs that the two methods agree have been published. In 1955, Parker [**4**] indicated a proof based on a simple relationship between the integrals of a function and its inverse. Later, Cable [**1**] gave a short proof using integration by parts. Both of these authors assumed that the function whose graph is revolved to form the boundary of the solid is monotone and has a continuous derivative. Recently Carlip [**2**] gave a proof that the disk and shell methods agree, assuming only continuity and monotonicity, based on the fundamental theorem of calculus and the principle that two functions with the same derivative that agree at a point are equal. In this note we show that Parker's original proof actually requires only these weaker hypotheses too.

Referring to Figure 1, we would expect that

$$\int_a^b h(x)\,dx + \int_{h(a)}^{h(b)} h^{-1}(y)\,dy = bh(b) - ah(a).$$

This relationship is the basis for a geometric interpretation of integration by parts, that occurs in Courant [**3**] and many other calculus texts. We sketch the formal proof below. I tell my students this is a "happy theorem" because we are able to formally prove a result that had better be true if our definition of integral successfully captures the notion of area.

Theorem 1. *Suppose that* $h\colon [a,b] \to [h(a), h(b)]$ *is monotone and continuous. Then* $\int_{h(a)}^{h(b)} h^{-1}(y)\,dy = bh(b) - ah(a) - \int_a^b h(x)\,dx$.

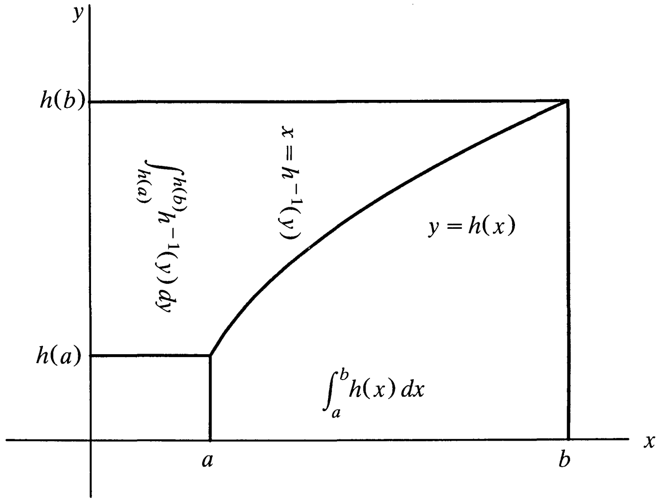

Figure 1

Proof. Observe that the function h gives a one-to-one correspondence between partitions of $[a, b]$ and partitions of $[h(a), h(b)]$. Furthermore, if h is increasing, every upper sum for $I_1 = \int_a^b h(x)\,dx$ corresponds to a lower sum for $I_2 = \int_{h(a)}^{h(b)} h^{-1}(y)\,dy$, and the sum of the two is $bh(b) - ah(a)$ (see Figure 2). Hence the supremum over all partitions of $[h(a), h(b)]$ of the lower sums for I_2 must be $bh(b) - ah(a)$ minus the infimum of the upper sums for I_1 over all partitions of $[a, b]$. Thus it follows from the definition of the integral that $I_2 = bh(b) - ah(a) - I_1$. A similar argument applies if h is decreasing. (In this case, of course, the image of $[a, b]$ under h should properly be written $[h(b), h(a)]$. But the integral formula is identical in either case.)

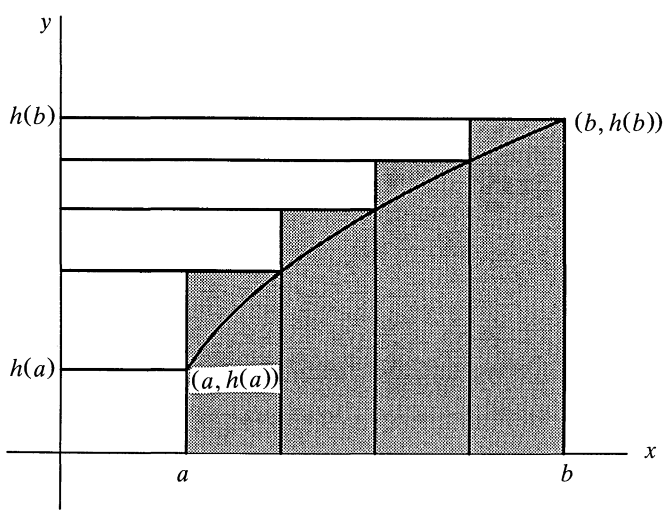

Figure 2
Riemann sums for $\int_a^b h(x)\,dx$ and $\int_{h(a)}^{h(b)} h^{-1}(y)\,dy$.

As an example, we apply this theorem to evaluate $\int_0^1 \arcsin(y)\,dy$. (Note that the usual approach via integration by parts leads to an improper integral.)

$$\int_0^1 \arcsin(y)\,dy = \frac{\pi}{2} - \int_0^{\pi/2} \sin(x)\,dx = \frac{\pi}{2} - 1$$

We now show that the disk and shell methods agree. The following theorem covers a typical case.

Theorem 2. *Suppose that $f\colon [a,b] \to [f(a), f(b)]$ is continuous and increasing. Let R be the region bounded by the x-axis, the lines $x = a$, $x = b$, and the graph of $y = f(x)$. Let S be the solid obtained by revolving R around the x-axis. Then the shell and disk methods for computing the volume of S give the same result.*

Proof. The shell method gives (make a sketch!)

$$\begin{aligned} VBS &= 2\pi\int_0^{f(a)} y[b-a]\,dy + 2\pi\int_{f(a)}^{f(b)} y\left[b - f^{-1}(y)\right]dy \\ &= \pi(b-a)f(a)^2 + \pi\left[b\left(f(b)^2 - f(a)^2\right) - \int_{f(a)}^{f(b)} 2yf^{-1}(y)\,dy\right] \\ &= \pi\left[bf(b)^2 - af(a)^2 - \int_{f(a)}^{f(b)} 2yf^{-1}(y)\,dy\right] \end{aligned}$$

The disk method says the volume is given by

$$VBD = \pi\int_a^b [f(x)]^2\,dx.$$

We apply Theorem 1 to the function $h\colon [a,b] \to [f(a)^2, f(b)^2]$ defined by $h(x) = [f(x)]^2$. We see $h^{-1}(y) = f^{-1}(\sqrt{y})$ and then make the substitution $t^2 = y$:

$$\begin{aligned} \int_a^b [f(x)]^2\,dx &= bf(b)^2 - af(a)^2 - \int_{f(a)^2}^{f(b)^2} f^{-1}\left(\sqrt{y}\right)dy \\ &= bf(b)^2 - af(a)^2 - \int_{f(a)}^{f(b)} f^{-1}(t)2t\,dt. \end{aligned}$$

Thus $VBD = VBS$.

Parker [4] shows how Theorem 1 can also be used to integrate any positive power of a monotone function by computing corresponding moments of its inverse function.

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