FALLACIES, FLAWS, AND FLIMFLAM

EDITOR

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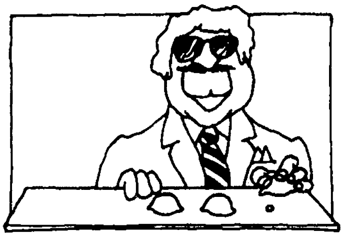


This column solicits mistakes, fallacies, howlers, anomalies and the like, that raise interesting mathematical issues and may be useful for teachers. Submissions need not be original provided a full reference is given (photocopies would be appreciated). Details of sources and comments about material already published are welcome. A black suit (♠ and ♣) in the text is a suggestion that the reader stop and consider the item before reading further; a red suit (♡ and ◇) indicates that the reader can proceed; a major suit (♠ or ♡) indicates the end of a proof or a solution.

FFF#178. Not many real sets.

Theorem. The cardinality of the class of subsets of the closed unit interval does not exceed the cardinality of the closed unit interval itself.

Proof. Let S be a subset of $[0, 1]$. We assign to S an element of $[0, 1]$, written in binary form as follows. If S is void, assign to S the number $0.00000\ldots$. Otherwise, the first binary digit after the decimal point of the corresponding number is 1. If S has nonvoid intersection with the interval $[0, 1/2]$, let the second binary digit of the corresponding number be 1; otherwise, assign 0. If S has nonvoid intersection with the interval $[1/2, 1]$, let the third binary digit be 1; otherwise, assign 0.

In a similar way we assign the next four digits of the number corresponding to S, according as to whether S intersects the four equal intervals into which $[0, 1]$ is partitioned, and then the next eight digits, and so on.

Each point in S is uniquely determined as the intersection of a nested sequence of intervals of the form $[(i-1)2^{-j}, i2^{-j}]$, and so is tracked by a 1 in the appropriate places of the binary expansion of the number corresponding to S, so we obtain a correspondence between the class of sets S and some subset of $[0, 1]$. This yields the desired result.♡

FFF#179. A wrong version of Stirling's formula.

Keith Brandt of Rockhurst University in Kansas City, MO invites students and teachers to look more closely at the limits underlying statements of the form $f(n) \approx g(n)$. He uses Stirling's formula as a case in point. This holds that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\cdot n^n \cdot e^{-n}} = 1,$$

so that, for large n, we may assume $n! \approx \sqrt{2\pi n} \cdot n^n \cdot e^{-n}$. A colleague of his, a chemist, once showed him how to use Stirling's formula and logarithms to derive a weaker result known as Stirling's approximation (often used in statistical mechanics). This is derived as follows.

Apply the logarithm to both sides of Stirling's formula to obtain

$$\ln(n!) \approx \frac{1}{2}\ln(2\pi n) + n\ln n - n.$$

The term $\frac{1}{2}\ln(2\pi n)$ is then declared "insignificant", and neglected to yield Stirling's approximation

$$\ln n! = n\ln n - n. \tag{1}$$

While this argument may be intuitive, it is not very satisfying from a mathematician's point of view. First, Stirling's approximation can be derived from scratch using only elementary calculus. Use Riemann sums to show that

$$\int_1^n \ln x\,dx \le \ln n! \le \int_2^{n+1} \ln x\,dx.$$

Evaluate the integrals, divide by $n\ln n - n$ and take limits. Second, the connection between Stirling's formula and Stirling's approximation is a bit subtle, and deserves a closer look.

The chemist's approach relies on the following two results, the first of which justifies applying the logarithm to both sides of Stirling's formula.

Lemma 1. *Suppose*

$$\lim_{n\to\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{n\to\infty} g(n) = \infty.$$

If $\lim_{n\to\infty} f(n)/g(n) = K > 0$, *then* $\lim_{n\to\infty}(\ln f(n))/(\ln g(n)) = 1$. *In particular, if* $f(n) \approx g(n)$, *then* $\ln f(n) \approx \ln g(n)$.

Lemma 2. $\lim_{n\to\infty}(\ln n)/(\ln n!) = 0$.

Lemma 1 is a good exercise for students in advanced calculus and Lemma 2 for students in elementary calculus. Lemma 1 suggests a second question: can we exponentiate both sides of $f(n) \approx g(n)$? Doing so to Stirling's approximation (1) yields

$$n! \approx \frac{n^n}{e^n}.$$

This elegant looking formula is not valid, since $\lim_{n\to\infty} n!/(n^n e^{-n}) = +\infty$. It is however mentioned in the literature (see [**1**, p. 24] or [**2**, p. 263]). Of course, ter Haar immediately applies the wrong formula to an expression involving the logarithm of $n!$. Franklin goes on to sharpen his "rough estimate" until he reaches Stirling's formula. How many students are aware of Lemma 1? How many are aware that there is no similar result for exponentiation?

References

1. ter Haar, D., *Elements of statistical mechanics*. Rinehart, 1954.
2. Frankin, P. *Methods of advanced calculus*. McGraw-Hill, 1944.

FFF#180. Integration discrepancies.

Every once in a while, we run an item where neglect or misuse of the constant of integration leads to some strange results. This often prompts correspondence from readers with their own favorite examples. Here are some more.

Roger B. Nelsen of Lewis & Clark College in Portland, OR found these answers to an integration when he was grading an examination.

Problem. Integrate

$$\int \frac{1}{\sqrt{x-x^2}}\,dx.$$

Solution 1. Completing the square under the radical in the denominator yields

$$\int \frac{1}{\sqrt{x-x^2}}\,dx = \int \frac{2}{\sqrt{1-(2x-1)^2}}\,dx = \arcsin(2x-1)+C. \qquad \heartsuit$$

Solution 2. Factoring the denominator as $\sqrt{1-x}\sqrt{x}$ and substituting $u=\sqrt{x}$ yields

$$\int \frac{1}{\sqrt{x-x^2}}\,dx = 2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin(\sqrt{x})+C. \qquad \heartsuit$$

Ollie Nanyes of Bradley University in Peoria, IL draws attention to the fact that, depending on the substitution, one obtains variously

$$\int \cosh x \sinh x = \frac{1}{2}\cosh^2 x + C$$

and

$$\int \cosh x \sinh x = \frac{1}{2}\sinh^2 x + C.$$

However, the issues are more subtle in the following:

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \frac{\cot x + \csc x}{\cot x + \csc x}\,dx = \int \frac{\csc x \cot x + \csc^2 x}{\cot x + \csc x}\,dx \\ &= -\ln|\cos x + \cot x| + C\end{aligned}$$

$((k-1)\pi < x < k\pi)$, k an integer. We also have

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x}\,dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\,dx \\ &= \ln|\csc x - \cot x| + C\end{aligned}$$

$((k-1)\pi < x < k\pi)$, k an integer. Note that the absolute value signs are essential to the solution.

David M. Bloom of Brooklyn College in New York, NY, reports that he has shown his class alternative indefinite integrals of the secant function. A table of values gives $\int \sec x = \ln|\sec x + \tan x| + C$, while writing the integrand as

$$\sec x = \frac{\cos x}{1 - \sin^2 x}$$

and making the substitution $u = \sin x$ leads, by a standard partial fraction evolution to the answer

$$\frac{1}{2}(\ln|1+u| - \ln|1-u|) = \frac{1}{2}(\ln|1+\sin x| - \ln|1-\sin x|) + C.$$

Here the constants C are actually the same in both expressions, as can be verified using a little trigonometry.

FFF#181. Finding asymptotes.

Carl Libis of Richard Stockton College in Pomona, NJ sends in a student solution to the exercise of finding all horizontal asymptotes for

$$f(x) = \frac{2}{x-3} - \frac{x}{x+2}.$$

$$\begin{aligned} f(x) &= \frac{2}{x-3} - \frac{x}{x+2} = \frac{2(x+2)}{(x-3)(x+2)} - \frac{x}{x+2} \\ &= \frac{2x+4}{x^2-x-6} + \frac{-x}{x+2} = \frac{x+4}{x^2-4} \end{aligned}$$

(presumably by adding the numerators and the denominators). Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+4}{x^2-4} = \frac{0+4}{0-4} = -1,$$

so the horizontal asymptote is $y = -1$ (which is correct).

COMMENTS ON PREVIOUS FFFs

David Cantrell of Tuscaloosa, AL has comments on two of the items appearing in the issue of January, 2001. In FFF#167 (*CMJ* **32**, 48), an increase from 0 to 1 was reported as one of 100%. Cantrell says, "I could not help but wonder what they would have listed as 'Pct. change from previous week' if there had been 0 'previous week's jokes' for some politician and some positive number of jokes about him for the earlier week. You may think this is somewhat frivolous, but it reminded me of a thread from last December in sci.math concerning an actual court case about whether an environmental impact study was mandated. You might be interested to take a look at http://forum.swarthmore.edu/epigone/sci.math/petwahpoi."

In the same issue, FFF#173 referred to a problem in a textbook about Helen of Troy, and what would be necessary to launch five ships. To my query about the intent of the author, Cantrell responds, "Is there whimsy here? Surely! Apparently, you are unaware of the joke, long a favorite of mine, which is well known, especially in the scientific community. Just do a *Google* search for 'millihelen'. You will find many references. From one of them, I extracted the following:

> 1 *Helen* (H) is a huge unit of beauty, and is thus inconvenient to work with. The smaller unit, the milliHelen (mH), representing the amount of beauty to launch a single ship is more workable."

SOURCES

CHANCE News 9.08 for July 3, 2000 to August 9, 2000, draws attention to an article by Jim Holt in *Lingua Franca* (July/August, 2000) http://www.linguafranca.com/0007/hypo.html in which the author makes the following comment:

> From Gödel's incompleteness theorem we know that in any formal system of arithmetic there are infinitely many propositions that are neither provable nor disprovable. Could Goldbach's conjecture be one of them? (That is what Uncle Petros [in the book by A.K. Doxiadis] begins to suspect.) If Goldbach's conjecture could be shown to be undecidable—neither provable nor disprovable—then this would be tantamount to proving it true! For if it was false, there must be some counterexample to it. But such a counterexample would constitute a disproof of the conjecture—thereby contradicting its undecidability.

An interesting article by Christopher Essex, Matt Davison and Christian Schulzky entitled *Numerical monsters* appears in *SIGSAM Bulletin* **134** (December: 2000) 16–32. In this article, graphs of various functions are plotted using *Maple* or *Matlab*, with results that are exceedingly perverse. Readers with suitable software might wish to plot the function $T(x) = e^x \ln(1 + e^{-x})$, and see what effect the machine "epsilon" has on the graph when e^{-x} becomes sufficiently small. The graph resembles a sequence of "tiger stripes". A second example concerns a simple 2×2 matrix system

$$\begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+\delta \\ 2 \end{pmatrix}.$$

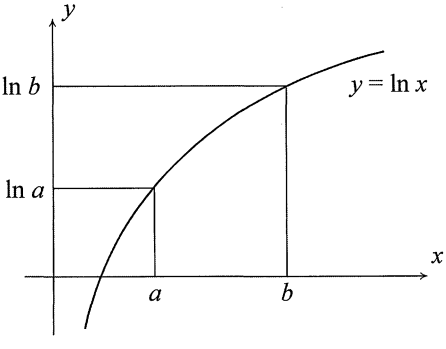
whose solution is $(x_1, x_2) = (1, 1)$. However, solving the system in stages determined by expressing the matrix as a product of a lower- and an upper-triangular matrix, and plotting the solution for x_1 as a function of δ leads to more tiger stripes. Similar round-off problems upset the graph of $x^{-1}(\sqrt{x+9} - 3)$ when x is close to 0. A "bow tie monster" is the result of having the computer plot the difference between $(x-1)^2$ and $x^2 - 2x + 1$ in terms of $x - 1$ or plot the difference between $\sin x$ and its cubic Taylor polynomial $x - (1/6)x^3$ near $x = 0$. These and other examples are analyzed by the authors, along with illustrations of the printouts. They conclude:

> The objective ... was to introduce a collection of simple calculations which are certain to create numerical mayhem in nearly any contemporary system in its plotting environment.... The visual environment is becoming more and more important in practical applications for computational programs. It is also the great leveler between computer algebra environments and floating point ones, as all numbers, no matter how precise, must still be jammed into a not so large number of pixels. How these pixels are filled leads to the most idiosyncratic software

arithmetics, as they are designed to meet the harsh demands of authentic mathematics in artificial graphical environments.... The goal is to *understand* and *recognize* the numerical idiosyncracies. They will not be so easily recognized in real applications as they are in these test cases, even in a case where they may be just as significant. Moreover they can never be fully eliminated from finite computers.... There is much more to say about these monsters in future publications. It is hoped that more monsters will be collected, not only because of their practical value but also because they are fun.

Mathematics Without Words

Roger Nelsen (Lewis & Clark College, nelsen@lclark.edu) shows how to integrate the natural logarithm:



$$\int_a^b \ln x\,dx = b\ln b - a\ln a - \int_{\ln a}^{\ln b} e^y\,dy = x\ln x\Big|_a^b - (b-a) = (x\ln x - x)\Big|_a^b$$