Differentiability of Exponential Functions

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We present a new proof of the differentiability of exponential functions. It is based entirely on methods of differential calculus. No current or recent calculus text gives or cites a proof of the differentiability that depends only on such elementary tools. Our proof makes it possible to give a comprehensive treatment of the derivative properties of exponential and logarithmic functions in that order in differential calculus, building on the standard introduction to these topics in precalculus courses. This is the logical order and has considerable pedagogical merit.

Most calculus books defer the treatment of exponential and logarithmic functions to integral calculus in order to prove differentiability. A few texts introduce these topics in differential calculus under the heading of "early transcendentals" but defer the proof of differentiability to integral calculus. Both approaches have serious pedagogical faults, which are discussed later in this paper.

Our proof that exponential functions are differentiable provides the missing link that legitimizes the "early transcendentals" presentation.

Preliminaries

We assume that a^r has been defined for $a > 0$ and r rational in a precalculus course and that the familiar rules of exponents are known to hold for rational exponents. It is natural to define a^x for $a > 0$ and x irrational as the limit of a^r as $r \to x$ through the rationals. In this way, a^x is defined for all real x.

Basic properties of a^x for real x are inherited by limit passages from corresponding properties of a^r for r rational. These properties include the rules of exponents with real exponents and

$$a^x \text{ is positive and continuous,}$$
$$a^x \text{ is increasing if } a > 1,$$
$$a^x \text{ is decreasing if } a < 1.$$

It is not especially difficult to justify the definition of a^x for x irrational and to derive the foregoing properties of a^x for x real, but there are a lot of small steps. A program along these lines is carried out by Courant in [**2**, pp. 69–70]. The general idea of each step is well within the grasp of students in typical calculus classes. However, just as properties of a^r with r rational are routinely stated without proof, it is better to give just an overview of the basic properties of a^x with x real, illustrated with graphs, and move on to the question of differentiability, which is more central to differential calculus.

A more complete development, beginning with the derivation of properties of a^r with r rational, might be given in an honors class. The properties can be extended to a^x with x real with the aid of the density of the rationals in the reals and the squeeze laws for limits. The conclusion that a^x with $a > 1$ is increasing also relies on the following proposition which should seem evident from graphical considerations:

If f is a continuous function on a real interval I
and f is increasing on the rational numbers in I,
then f is increasing on I.

The same proposition will provide a key step in the proof that a^x is differentiable.

Henceforth, we restrict our attention to properties of a^x with $a > 1$. Corresponding properties of a^x with $0 < a < 1$ follow from $a^x = (1/a)^{-x}$.

The differentiability of a^x

Consider an exponential function a^x with any $a > 1$. In order to prove that a^x is differentiable for all x, the main task is to prove that it is differentiable at $x = 0$. Our proof of this depends only on methods of differential calculus. It is motivated by the fact that the graph of a^x (see Figure 1) is concave up, even though this fact is not assumed *a priori.*

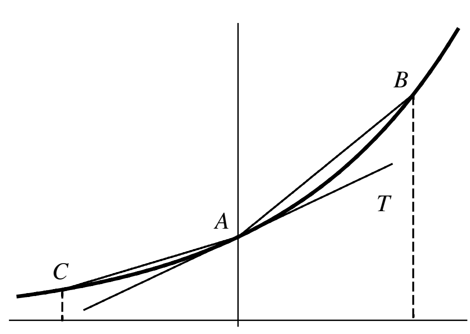

Figure 1. Graph of a^x with $B = (x, a^x)$ and $C = \left(-x, a^{-x}\right)$ for $x > 0$.

In Figure 1, imagine that $x \to 0$ with $x > 0$ and x decreasing. Then B and C slide along the curve toward A. The upward bending of the curve seems to imply that

$$\text{slope } \overline{AB} \text{ decreases,} \quad \text{slope } \overline{AC} \text{ increases,}$$

$$\text{and} \quad \text{slope } \overline{AB} - \text{slope } \overline{AC} \to 0.$$

It follows that the slopes of $\overline{AB}$ and $\overline{AC}$ approach a common limit, which is the slope of the tangent line T in Figure 1 and the derivative of $f(x) = a^x$ at $x = 0$. This geometric argument will be made rigorous.

The curve in Figure 1 is actually the graph of $f(x) = 2^x$. The following table gives values of the slopes of $\overline{AB}$ and $\overline{AC}$ rounded off to two decimal places. It appears that the slopes of $\overline{AB}$ and $\overline{AC}$ approach a common limit, which is $f'(0) = \text{slope } T \approx 0.7$.

x	1	1/2	1/4	1/8	1/16	1/32
slope $\overline{AB}$	1	.83	.76	.72	.71	.70
slope $\overline{AC}$	.50	.59	.64	.66	.67	.69

With this preparation, we are ready to prove that $f(x) = a^x$ is differentiable at $x = 0$. The foregoing geometric description of the proof and the numerical evidence should be informative and persuasive to students, even if they do not follow all the details of the argument.

Theorem 1. *Let $f(x) = a^x$ with any $a > 1$. Then f is differentiable at $x = 0$ and $f'(0) > 0$.*

Proof. To express our geometric observations in analytic terms, let

$$m(x) = \frac{f(x) - f(0)}{x - 0} = \frac{a^x - 1}{x}.$$

In Figure 1, $x > 0$ and

$$\text{slope } \overline{AB} = m(x),$$

$$\text{slope } \overline{AC} = m(-x).$$

We shall prove that, as $x \to 0$ with $x > 0$ and x decreasing, $m(x)$ and $m(-x)$ approach a common limit, which is $f'(0)$.

To begin with, $m(x)$ is continuous because a^x is continuous. The crux of the proof, and the only tricky part, is to show that

$$m(x) \text{ is increasing on } (0, \infty) \text{ and } (-\infty, 0).$$

We give the proof only for $(0, \infty)$ since the proof for $(-\infty, 0)$ is essentially the same. We show first that m is increasing on the rationals in $(0, \infty)$. Fix rational numbers r and s with $0 < r < s$ and let a vary with $a \geq 1$. Define

$$g(a) = m(s) - m(r) = \frac{a^s - 1}{s} - \frac{a^r - 1}{r}.$$

Then $g(a)$ is continuous for $a \geq 1$ and

$$g'(a) = a^{s-1} - a^{r-1} > 0 \quad \text{for } a > 1.$$

Thus, $g(a)$ increases as a increases and $g(a) > g(1) = 0$ for $a > 1$, so

$$m(r) < m(s) \quad \text{for } 0 < r < s.$$

Thus, $m(x)$ is continuous on $(0, \infty)$ and $m(x)$ increases on the rational numbers in $(0, \infty)$. As noted earlier, this implies that $m(x)$ increases on $(0, \infty)$. The argument for the interval $(-\infty, 0)$ is similar.

For $x > 0$,

$$\begin{aligned}
&m(-x) = m(x)a^{-x}, \\
&0 < m(-x) < m(x), \\
&0 < m(x) - m(-x) = m(x)\left(1 - a^{-x}\right).
\end{aligned}$$

Let $x \to 0$ with x decreasing. Then

$$m(x) \text{ decreases}, \quad m(-x) \text{ increases}, \quad m(x) - m(-x) \to 0.$$

It follows that $m(x)$ and $m(-x)$ approach a common limit as $x \to 0$, which is $f'(0)$. Furthermore, $0 < m(-x) < f'(0) < m(x)$ for $x > 0$, which implies that $f'(0) > 0$. ∎

We believe that this proof is new. We have been unable to find any other proof that depends only on methods of differential calculus. However, the interplay between convexity and differentiability has a long history, and we recommend Chapter 1 of Artin [**1**] to interested readers.

Theorem 1 and familiar reasoning give the principal result on the differentiability of exponential functions.

Theorem 2. *Let $f(x) = a^x$ with any $a > 1$. Then f is differentiable for all x and $f'(x) = f'(0)a^x$.*

It follows that $f''(x) = f'(0)^2 a^x > 0$, and $f(x) = a^x$ is concave up, as anticipated. By routine arguments,

$$\begin{aligned}
a^x &\to \infty \quad \text{as} \quad x \to \infty, \\
a^x &\to 0 \quad \text{as} \quad x \to -\infty.
\end{aligned}$$

The intermediate value theorem implies that the range of a^x is $(0, \infty)$.

The natural exponential function e^x

The next step in the development of properties of derivatives of exponential functions is to define e, the base of the natural exponential function e^x, within the context of differential calculus. Different authors define e in various ways. Some of the definitions involve more advanced concepts. We prefer a definition of e based on an important property of e^x, namely that e is the unique number for which

$$\left.\frac{d}{dx}e^x\right|_{x=0} = 1.$$

In view of Theorem 2, an equivalent property is

$$\frac{d}{dx}e^x = e^x \quad \text{for all } x.$$

It is not difficult to justify the definition of e and, at the same time, to find an explicit formula for e. To begin with, consider any base $a > 1$. Since 2^x is increasing and has range $(0, \infty)$, there is a unique number $c > 0$ such that $a = 2^c$. By Theorem 1,

$$\left.\frac{d}{dx}a^x\right|_{x=0} = \lim_{x\to 0}\frac{a^x - 1}{x} = c\lim_{cx\to 0}\frac{2^{cx} - 1}{cx} = cm,$$

$$\text{where} \quad a = 2^c \quad \text{and} \quad m = \left.\frac{d}{dx}2^x\right|_{x=0} = \lim_{x\to 0}\frac{2^x - 1}{x}.$$

Observe that

$$\left.\frac{d}{dx}a^x\right|_{x=0} = 1 \text{ only for } c = 1/m \text{ and } a = 2^{1/m}.$$

Therefore, $e = 2^{1/m}$. Since $m \approx 0.7$ by previous calculations, $e \approx 2.7$. Of course, there are much better approximations for e.

Since e is one particular value of a, the function e^x has all the properties mentioned earlier for general exponential functions a^x with $a > 1$. Thus, e^x is increasing and concave up,

$$e^x \to \infty \quad \text{as} \quad x \to \infty,$$
$$e^x \to 0 \quad \text{as} \quad x \to -\infty,$$

and the range of e^x is $(0, \infty)$.

With this foundation, all relevant applications of exponential functions become available in differential calculus.

Logarithmic functions

Once the basic properties of exponential functions have been established, it is easy to introduce logarithmic functions as corresponding inverse functions and to develop their relevant properties within differential calculus.

The natural logarithmic function (or natural log) is defined by

$$y = \ln x \iff x = e^y.$$

The derivative rule for inverse functions implies that $y = \ln x$ is differentiable and

$$\frac{d}{dx}\ln x = \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus,

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

The familiar algebraic properties and asymptotic properties of logarithmic functions follow easily from corresponding algebraic rules of exponents and asymptotic properties of exponential functions. In typical textbooks that defer exponential and logarithmic functions to integral calculus, proofs of algebraic properties of $\ln x$ are based on the uniqueness of solutions to initial value problems and are less informative for most first-year calculus students.

Comparisons

It is worthwhile to contrast our approach with current practices. Most mainstream calculus texts, such as [**5**] and [**8**], defer the entire discussion of exponential and logarithmic functions to integral calculus, where exponential functions are expressed as inverses of logarithmic functions in order to establish their differentiability. As we wrote earlier, this is the reverse of the natural order. It has the unfortunate consequence that exponential functions are often defined in two different ways that ultimately have to be reconciled. The upshot is a circuitous argument that blurs the distinction between definitions and conclusions. Moreover, a substantial block of material about derivatives and rates of change is presented in integral calculus, instead of in its natural place in differential calculus. Exponential and logarithmic functions have many important applications, such as motion with resistance, that belong in differential calculus.

A few books, often called "early transcendentals" texts, introduce exponential and logarithmic functions in differential calculus. Although this arrangement is an improvement over the standard approach, the differentiability of exponential functions remains a stumbling block. Some of these books, such as [**3**] and [**4**], simply display a little numerical evidence and/or a plausibility argument in support of differentiability and then assume differentiability thereafter. Others, such as [**6**] and [**7**], start out with plausibility arguments in differential calculus and then return to the subject in integral calculus where proofs are given. Neither alternative is really satisfactory. It is much better, both logically and pedagogically, to settle the question of differentiability when the issue arises.

Our proof that exponential functions are differentiable makes it possible to give a mathematically complete "early transcendentals" presentation of exponential and logarithmic functions in differential calculus. Later on, when methods of integral calculus are applied to exponential and logarithmic functions, progress will not be impeded by unfinished business in differential calculus.

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