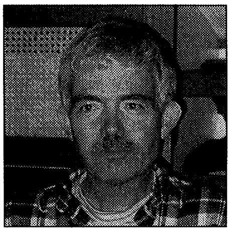
The Brahmagupta Triangles

Raymond A. Beauregard and E. R. Suryanarayan



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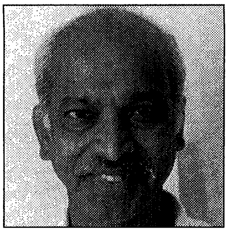

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The study of geometric objects has been a catalyst in the development of number theory. For example, the figurate numbers (triangular, square, pentagonal, ...) were a source of many early results in this field [**4**]. Measuring the length of a diagonal of a rectangle led to the problem of approximating $\sqrt{N}$ for a natural number N. The study of triangles has been of particular significance.

Heron of Alexandria (c. A.D. 75) gave the well-known formula for the area Δ of a triangle in terms of its sides: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ is the semiperimeter of the triangle having sides a, b, c [**4**]. He illustrated this with a triangle whose sides are $13, 14, 15$ and whose area is 84. A triangle with integer sides and area is called a *heronian triangle* in his honor.

The Indian astronomer and mathematician Brahmagupta (born A.D. 598)[1] analyzed the class of heronian triangles having *consecutive* integer sides—what we now call *Brahmagupta triangles*. Brahmagupta denoted the first eight such triangles as the triples $(3,4,5)$, $(13,14,15)$, $(51,52,53)$, $(193,194,195)$, $(723,724,725)$, $(2701,2702,2703)$, $(10083,10084,10085)$, $(37633,37634,37635)$ [**8**]. How did he compute these values?

Generating the triangles. We will show how all such triangles can be determined, using computations analogous to Brahmagupta's. Let us assume that the consecutive sides of a Brahmagupta triangle are $t-1, t, t+1$, where t is a positive integer; see Figure 1 when $t > 4$. Then its semiperimeter is $s = 3t/2$, and by Heron's formula its area is

$$\Delta = \frac{t}{2}\sqrt{3\left[\left(\frac{t}{2}\right)^2 - 1\right]}. \tag{1}$$

[1]This paper commemorates Brahmagupta's fourteenth centenary.

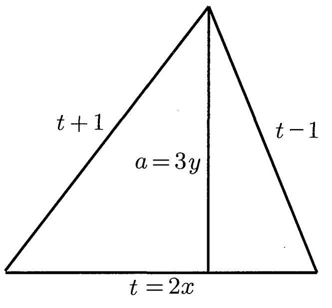

Figure 1. A Brahmagupta triangle.

When we impose the additional condition that Δ must also be an integer, then the base t of the triangle must be even and the altitude a of the triangle must be an integer multiple of 3. To see this, write (1) as $16\Delta^2 = 3t^2\left(t^2 - 4\right)$. Since $\Delta = at/2$, this equation reduces to

$$4a^2 = 3\left(t^2 - 4\right). \tag{2}$$

If t were odd then the factors on the right side of (2) would all be odd, and this is not possible. Thus t is even and we may write $t = 2x$ (where x is an integer). The area of the triangle in Figure 1 is then $\Delta = ax$, and it follows that a is a rational number. But

$$a^2 = 3(x^2 - 1), \tag{3}$$

which is an integer, so a itself must be an integer and hence a multiple of 3. Writing $a = 3y$, we reduce (3) to the so-called *Pell equation*

$$x^2 - 3y^2 = 1. \tag{4}$$

Although the (integer) solutions to the Pell equation are well known, we will derive them easily using matrix methods. Equation (4) can be expressed by the requirement that the matrix $B = \begin{bmatrix} x & y \\ 3y & x \end{bmatrix}$ be unimodular (that is, $\det B = 1$). Let $x_1 = 2$ and $y_1 = 1$, since these values describe the first Brahmagupta triangle—and provide the first positive solution to (4). If we define x_n and y_n by

$$\begin{bmatrix} x_1 & y_1 \\ 3y_1 & x_1 \end{bmatrix}^n = \begin{bmatrix} x_n & y_n \\ 3y_n & x_n \end{bmatrix}, \qquad \text{for} \quad n = 0, 1, 2, \ldots, \tag{5}$$

then unimodularity shows that x_n and y_n satisfy (4). (We take B^0 to be the identity matrix, so that $x_0 = 1$ and $y_0 = 0$.)

Working with (5) and the values $x_1 = 2$ and $y_1 = 1$, we obtain the following recurrence relations:

$$x_{n+1} = 2x_n + 3y_n, \quad y_{n+1} = 2y_n + x_n, \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Eliminating y_n from these two equations yields $2x_{n+1} - 3y_{n+1} = x_n$, or $3y_n = 2x_n - x_{n-1}$. Similarly, eliminating x_n yields $x_n = 2y_n - y_{n-1}$. Substituting these expressions into the recurrence relations, we find that x_n and y_n satisfy the difference equations

$$x_{n+1} = 4x_n - x_{n-1} \quad y_{n+1} = 4y_n - y_{n-1}. \tag{6}$$

These formulas were obtained by Aubry in 1911 [**4,** p. 200]. In 1879 Hoppe had obtained similar results while working with triangles with integer area whose sides form an arithmetic progression (see [**2**] for an analysis of these triangles). Thus we obtain the sequences

$$x_n = 2, 7, 26, 97, \ldots, \quad y_n = 1, 4, 15, 56, \ldots.$$

Diagonalizing the matrix B yields

$$\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} x + y\sqrt{3} & 0 \\ 0 & x - y\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{6}} \end{bmatrix}.$$

Computation with (5) and the values $x_1 = 2$ and $y_1 = 1$ then gives the closed forms

$$x_n = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n\right], \qquad y_n = \frac{1}{\sqrt{12}}\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right].$$

Note that $x_n + \sqrt{3}y_n = \left(x_1 + \sqrt{3}y_1\right)^n$.

Properties of the Brahmagupta triangles. The sequences

$$2x_n = 4, 14, 52, 194, \ldots, \quad 3y_n = 3, 12, 45, 168, \ldots$$

give the even side and the altitude, respectively, for the successive Brahmagupta triangles. The nth triangle has sides $2x_n$, $2x_n - 1$, $2x_n + 1$; semiperimeter $s = 3x_n$; and area $\Delta = 3x_n y_n$.

What is the length of the base segment MA from the median point M to the altitude point A as shown in Figure 2? The smaller right triangle has base length equal to the square root of

$$(2x_n - 1)^2 - 9y_n^2 = (2x_n - 1)^2 - 3(x_n^2 - 1) = (x_n - 2)^2.$$

Since the base of the nth Brahmagupta triangle is $2x_n$, the segment MA measures 2 (for any n)!

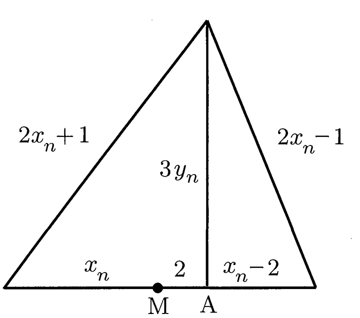

Figure 2. The nth Brahmagupta triangle.

We know that the area and altitude of the nth Brahmagupta triangle are integers. Many other terms associated with this triangle are rational numbers, including the altitudes extending from the other two sides, the radii of the circumcircle and the three excircles, the tangents of the vertex angles, and the radius of the associated 9-point circle [**3**], [**6**].

Properties of x_n and y_n. The sequences x_n and y_n have some interesting properties as well. For example, it can be shown that if m divides n, then x_m and y_m divide y_n. There is a nice analogy with the Fibonacci sequence F_n and the Lucas sequence L_n. Recall that if $F(t)$ and $L(t)$ are the generating functions of F_n and L_n/n, so that

$$F(t) = \sum_1^\infty F_n t^n, \quad L(t) = \sum_1^\infty \frac{L_n}{n} t^n,$$

then $F(t) = e^{L(t)}$ [**5**]. Similarly, if $X(t)$ and $Y(t)$ are the generating functions of x_n/n and y_n, then $Y(t) = te^{2X(t)}$. These and other properties of such sequences are discussed in [**7**].

More generally, if 3 is replaced by any square-free natural number N in (4), then the resulting sequences of integers x_n and y_n in (5) satisfy this Pell equation, and all solutions can be generated (as in the $N = 3$ case) from the least positive solution [**1**]. Brahmagupta knew that the ratios x_n/y_n converge very rapidly to $\sqrt{N}$. In fact, the present method is a faster way to compute rational approximations to $\sqrt{N}$ than computing successive convergents of the continued fraction expansion of $\sqrt{N}$: to compute the nth power of a matrix one need not compute all previous powers.

Our matrix technique shows that, in general, the two terms x_n and y_n satisfy $x_n + \sqrt{N}y_n = \left(x_1 + \sqrt{N}y_1\right)^n$. Did Brahmagupta find the triangles with consecutive integer sides and integer area in this way? Although this matrix approach might seem foreign to Brahmagupta's approach, in fact we took our cue from his work. He had discovered that solutions to Pell's equation $x^2 - Ny^2 = m$ obey a law of composition—his *samasa-bhavana*: If $x_1^2 - Ny_1^2 = m_1$ and $x_2^2 - Ny_2^2 = m_2$, then the pair $(x_1x_2 + Ny_1y_2,\ x_1y_2 + x_2y_1)$ satisfies $x^2 - Ny^2 = m_1m_2$ [**9**]. Now if we set $B(x,y) = \begin{bmatrix} x & y \\ Ny & x \end{bmatrix}$, the Pell equation $x^2 - Ny^2 = m$ becomes $\det\left(B(x,y)\right) = m$. Since

$$B(x_1,y_1)B(x_2,y_2) = B(x_1x_2 + Ny_1y_2,\ x_1y_2 + x_2y_1),$$

the *samasa-bhavana* is equivalent to the identity

$$\det\left(B(x_1,y_1)B(x_2,y_2)\right) = \det\left(B(x_1,y_1)\right) \cdot \det\left(B(x_2,y_2)\right).$$

Thus, Brahmagupta's law of composition incorporates both 2×2 matrix multiplication and the multiplicative property of determinants—ideas that were not formalized until the nineteenth century!

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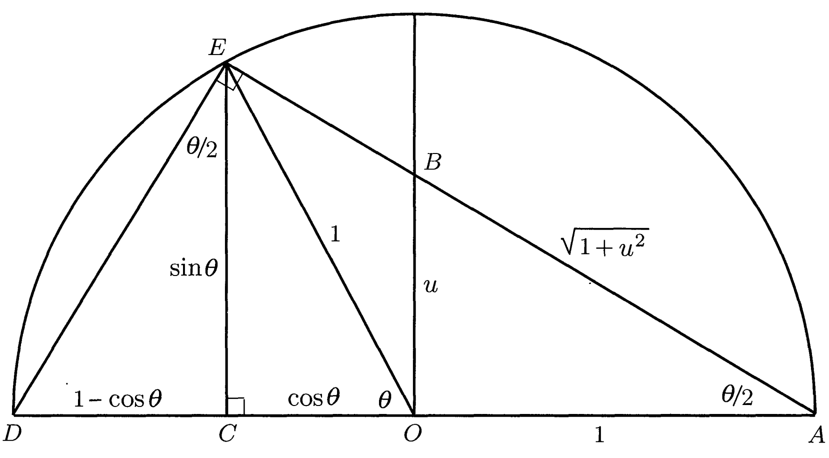
The Method of Last Resort (Weierstrass substitution)



$$u = \tan\frac{\theta}{2}$$

$$\overline{DE} = 2\sin\frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

$$\frac{\overline{CE}}{\overline{DE}} = \frac{\overline{OA}}{\overline{BA}} \quad \rightarrow \quad \sin\theta = \frac{2u}{1+u^2}$$

$$\frac{\overline{CD}}{\overline{DE}} = \frac{\overline{OB}}{\overline{BA}} \quad \rightarrow \quad \cos\theta = \frac{1-u^2}{1+u^2}.$$

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