A Fundamental Theorem of Calculus that Applies to All Riemann Integrable Functions

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The usual form of the Fundamental Theorem of Calculus is as follows:

**Theorem 1.** Let $f$ be Riemann integrable on $[a, b]$ and let $g$ be a function such that $g'(x) = f(x)$ on $[a, b]$. Then

$$\int_a^b f(x) \, dx = g(b) - g(a).$$

Unfortunately, this theorem only applies to Riemann integrable functions that are derivatives. Thus it cannot even be used to integrate the following simple function

$$f(x) = \begin{cases} 
0 & \text{if } -1 < x < 0 \\
1 & \text{if } 0 \leq x \leq 1.
\end{cases}$$

It is the purpose of this note to present a theorem that does apply to every integrable function. In stating our result we will need the following definitions.

**Definition 1.** The function $f: [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition if there exists $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x \text{ and } y \text{ in } [a, b].$$

**Definition 2.** A set $E$ of real numbers has measure zero if for each $\varepsilon > 0$ there is a finite or infinite sequence $\{I_n\}$ of open intervals covering $E$ and satisfying $\Sigma |I_n| \leq \varepsilon$ where $|I_n|$ is the length of $I_n$. If a property holds except on a set of measure zero, it is said to hold almost everywhere.

In [2] the author gave an elementary proof of the following result.

**Lemma.** If $f: [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition and $f'(x) = 0$ except on a set of measure zero, then $f$ is a constant function on $[a, b]$.

The proof required no measure theory other than the definition of a set of measure zero. This lemma was then used to prove that a bounded function that is continuous almost everywhere is Riemann integrable. We will use it here to establish our general form of the Fundamental Theorem of Calculus.

**Theorem 2.** Let $f$ be Riemann integrable on $[a, b]$ and let $g$ be a function that satisfies a Lipschitz condition and for which $g'(x) = f(x)$ almost everywhere. Then

$$\int_a^b f(x) \, dx = g(b) - g(a).$$

**Proof.** Let $F(x) = \int_a^x f(t) \, dt$. Since $f$ is bounded, $F$ satisfies a Lipschitz condition. From the fact that $f$ is continuous except on a set of measure zero (see [3] for an elementary proof), it follows that $F'(x) = f(x)$ almost everywhere. (This shows that every Riemann integrable function is almost everywhere the derivative of a function.
satisfying a Lipschitz condition.) It follows at once that
\[(F - g)'(x) = F'(x) - g'(x) = f(x) - f(x) = 0\]
almost everywhere. In addition \(F - g\) satisfies a Lipschitz condition. By the lemma there exists a real number \(k\) such that \(F(x) = g(x) + k\) on \([a, b]\). Setting \(x = a\) we have \(k = -g(a)\). Finally, setting \(x = b\), we get
\[\int_a^b f(x) \, dx = F(b) - g(a)\]
which completes the proof.

Note that Theorem 2 includes Theorem 1 since any function that has a bounded derivative satisfies a Lipschitz condition.

Let us now integrate the following function. Define
\[f(x) = \begin{cases} -x & \text{if } x \in S = \{1, 1/2, 1/3, \ldots\} \\ x^2 + 1 & \text{if } x \in [0, 1] \setminus S. \end{cases} \]
Since \(f\) is bounded and continuous except on \(S \cup \{0\}\), a set of measure zero, it is Riemann integrable. Let \(g(x) = x^3/3 + x\). Then \(g\) satisfies a Lipschitz condition and we have that \(g'(x) = x^2 + 1 = f(x)\) almost everywhere. Therefore,
\[\int_0^1 f(x) \, dx = g(1) - g(0) = 4/3.\]
In this case \(g'(x) \neq f(x)\) on an infinite set and yet Theorem 2 can still be used.

In closing, we give a useful corollary of Theorem 2.

**Corollary.** Let \(f\) be Riemann integrable on \([a, b]\) and let \(g\) be a continuous function such that \(g'(x) = f(x)\) except on a countable set. Then
\[\int_a^b f(x) \, dx = g(b) - g(a).\]

**Proof.** To use Theorem 2 we need only show that \(g\) satisfies a Lipschitz condition. Since \(f\) is integrable there exists \(M > 0\) such that \(|f(x)| \leq M\) for all \(x\) in \([a, b]\). Thus \(-M \leq g'(x) \leq M\) except on a countable subset of \([a, b]\). Let \(h(x) = Mx - g(x)\). Since \(h\) is continuous on \([a, b]\) and \(h'(x) = M - g'(x) > 0\) except on a countable set, it follows from a result in [1] that \(h\) is increasing on \([a, b]\). Thus for \(c\) and \(d\) in \([a, b]\) with \(c < d\) we have \(h(c) < h(d)\) which gives \(g(d) - g(c) \leq M(d - c)\). Similarly, we can show that \(-M(d - c) \leq g(d) - g(c)\) and therefore \(|g(d) - g(c)| \leq M(d - c)\). Thus \(g\) satisfies a Lipschitz condition and the proof follows immediately from Theorem 2.

**References**