Japanese University Entrance Examination Problems in Mathematics

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Mathematical Association of America

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Introduction

In Japan, higher education is considered to be one of the most important factors in obtaining a good job. Consequently, one's social status depends greatly on the level and quality of one's education. Since Japan is a fairly affluent nation, most families can afford higher education for their children. However, with a limited number of prestigious universities, competition on the entrance exam is very keen. Although the overall acceptance rate at four-year institutions in recent years has been above 60%, the acceptance rate at national and local public universities has been only 26%. As a result even at the elementary school level some Tokyo students attend additional evening or Saturday classes to prepare for the entrance examination for the best private junior high schools (grades 7, 8, 9) and high schools (grades 10, 11, 12) in order to be well prepared for the university entrance examination.

The purpose of this report is to give samples of Japanese entrance examinations in the field of mathematics along with the performance results of the 1990 examination. It is hoped that a better understanding of the level of mathematics expected of Japanese students will give mathematics educators in the U. S. a basis for comparison when reviewing the U. S. secondary school mathematics curriculum and expectation of student performance.

The editor deeply appreciates the generous assistance of Professor Hiroshi Fujita of Meiji University, Tokyo, Japan in obtaining the data. Professor Fujita is a member of the committee currently reviewing the Japanese national curriculum and is in charge of the Senior High School Mathematics Division of the Council for Educational Curriculum. He has presented papers on Mathematics Education of the Senior Secondary Level in Japan at the National Council of Teachers of Mathematics (NCTM) meetings in the U.S., at the International Congress on Mathematical Education (ICME) in Budapest, and at other international conferences. The editor also would like to express special thanks to Professor Richard Askey, University of Wisconsin, for his thorough and helpful reading of the manuscript and for his valuable comments and suggestions. The editor sincerely thanks the Alfred P. Sloan Foundation for the financial support which made this project possible. Also special appreciation is due Beverly Ruedi at the MAA for her efficient and excellent work. To the publisher, Obunsha, the author gratefully acknowledges their assistance and cooperation in granting permission to translate material from the "1991 Zenkoku Daigaku Nyuushi Mondai Seikai Suugaku (Kokkouritsu-daihen)" [1991 University Entrance Examinations in Mathematics: Problems & Solutions—Edition for National & Local Public Universities]. The editor attempted to give an accurate and readable translation, she accepts full responsibility for any errors that may appear here.

The entrance examination for a national or local public university in Japan consists of two parts:

1. Standardized Primary Exam—UECE

The University Entrance Center Examination (UECE) is administered by a government agency, the Center for University Entrance Examinations. This examination is based on the high school curriculum set forth by the Ministry of Education. Prior to 1990, this examination was called the Joint First Stage Achievement Test (JFSAT). All applicants for national or local public universities are required to take this examination. It is offered once a year, usually in mid-January, with the make-up examination scheduled one week later.

2. Secondary Exam

This additional exam is given independently by each university at a later date. Generally the examination dates for national and local public universities are divided into two types:

(i) Two-stage type—two exam dates, usually two weeks apart, with pre-determined allocation of entrants, about a 9 to 1 ratio.

(ii) AB-date (or ABC-date) type—two (or three) exam dates, A-date and B-date, (and C-date), usually one week apart, uniformly set each year.
A-date coincides with First-stage date in (i).

Thus there are four possible paths offering a student two chances to take an entrance exam for a public university:

(a) A-date exam ⟶ B-date exam
(b) A-date exam ⟶ Second-stage exam
(c) First-stage exam ⟶ B-date exam
(d) First-stage exam ⟶ Second-stage exam.

If a student passes the First-stage exam of a university and decides to enter the university, the student is required to proceed with registration shortly after the exam result is known. Then the student loses applicant status for the Second-stage exam or accepted status for the B-date exam, whichever status the student held.

Each university sets the relative weights of the two examination scores in its admissions decisions. The ratio of UECE score weights to the individual university examination weights is typically about 40% to 60%, or 50% to 50%. Tokyo University is an extreme case, setting the ratio of weights at 20% for the UECE score to 80% for the Tokyo University score.

In 1990, about 6% of those admitted to the four-year national or local public universities gained admission by recommendations; some took only the UECE and some were exempt from both the UECE and the individual university examination. Although the recent improvement in secondary examination scheduling enables a student to take two national or local public university exams, most applicants target an exam given on the first date or A-date because an exam given on the second date tends to be more difficult and has a smaller allocation of admittees.

According to the Ministry of Education, about 73% of university students were enrolled in private universities in 1987. In the past, JFSAT was primarily for the national and local public universities and only the individual university examination was required for private universities. When UECE replaced JFSAT in 1990, one of the objectives of the UECE Committee was to make the examination easier for private as well as public universities to utilize. Sixteen of the 342 private universities joined all 132 national and local public universities to participate in 1990. Twenty-one private universities participated in UECE in 1991 and 31 participated in 1992. Currently, most private university applicants do not take the UECE.

Private universities admit a somewhat larger percentage of their students by recommendations. In 1984 the rate was about 20%. In 1991 at Asia University in Tokyo, 40 of the 400 admitted by recommendations were exempt from taking the UECE and the university's entrance examination. Unlike national and local public universities, private universities' entrance examination dates are not uniform, allowing applicants to take many private university examinations. In recent years students have applied to an average of 6 private universities.

Even though the use of UECE in Japan is not yet as widespread as that of the SAT in the U.S., it plays a similar role. See the following table for comparison data.

1987	Japan*	U. S.**
Population age 18	1,883,000	3,667,000
High school graduates	1,792,000	2,647,000
Proportion of the age group	95%	72%
High school graduates enrolled in college	681,000	1,503,000
Proportion of HS grads	38%	57%
Proportion of the age group	36%	41%
Number of students who took:		
UECE (formerly JFSAT)	256,000[(1)]	
SAT		1,134,000
Proportion of HS grads enrolled in college	38%	75%
Proportion of HS grads	14%	43%
Proportion of the age group	14%	31%
Applicants to universities or junior colleges	1,025,000	
Accepted applicants	681,000	2,246,000[(2)]
Acceptance rate	66%	

The 1990 UECE in mathematics is divided into two sections: Section A consists of Mathematics I and Section B consists of either Mathematics II, Industrial Mathematics, or Accounting/Statistics I, II. Depending on the student's area of study, he or she will select the appropriate examination in Section B. Of the 327,543 applicants who took Section B in 1990, 327,034 took the Mathematics II, 52 took Industrial Mathematics, and 457 took Accounting/Statistics I, II. Because of the significantly greater number of applicants taking Mathematics I and II, only those two examinations will be considered in the following article.

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* Data from Monbusho: Outline of Education in Japan, 1989.
** Data from the *1989–90 Fact Book on Higher Education* by American Council on Education and Macmillan Co. and the *Digest of Education Statistics* (Government Printing Office, Washington, DC, 1991).
(1) The figure excludes approximately 138,000 of those who graduated prior to 1987.
(2) First time freshman including H.S. graduates of other years.

1990 UECE in Mathematics

Directions: Each problem contains several blanks. Blanks are represented by bracketed, underlined numbers. Each blank must be filled with a single digit or sign. See the method shown in the following examples and answer in the specified space on the answer sheet:

1. $\{\underline{1}\}, \{\underline{2}\}, \{\underline{3}\}, \{\underline{4}\}, \ldots$ each represent values between 0 and 9 or $+$ or $-$ signs. For example, to indicate -8 as the answer to $\{\underline{1}\}\{\underline{2}\}$, mark

$\{\underline{1}\}$ ⊖ + 0 1 2 3 4 5 6 7 8 9

$\{\underline{2}\}$ − + 0 1 2 3 4 5 6 7 ⑧ 9

2. If the answer is a fraction, reduce the fraction to its lowest terms and indicate the sign in the numerator. For example, to indicate $-2/9$ as the answer to $\{\underline{3}\}\{\underline{4}\}/\{\underline{5}\}$, mark

$\{\underline{3}\}$ ⊖ + 0 1 2 3 4 5 6 7 8 9

$\{\underline{4}\}$ − + 0 1 ② 3 4 5 6 7 8 9

$\{\underline{5}\}$ − + 0 1 2 3 4 5 6 7 8 ⑨

Mathematics A [Mathematics I] (100 points, 60 minutes)

Section 1 (30 points)

1. Suppose the polynomial $P(x)$ with integer coefficients satisfies the following conditions:

(A) If $P(x)$ is divided by $x^2 - 4x + 3$, the remainder is $65x - 68$.

(B) If $P(x)$ is divided by $x^2 + 6x - 7$, the remainder is $-5x + a$.

Then we know that $a = \{\underline{1}\}$.

Let us find the remainder $bx + c$ when $P(x)$ is divided by $x^2 + 4x - 21$.

Condition (A) implies that $\{\underline{2}\}\ b + c = \{\underline{3}\}\{\underline{4}\}\{\underline{5}\}$ and $a = \{\underline{1}\}$. Condition (B) implies that $\{\underline{6}\}\{\underline{7}\}\ b + c = \{\underline{8}\}\{\underline{9}\}$. It follows that $b = \{\underline{10}\}$ and $c = \{\underline{11}\}\{\underline{12}\}\{\underline{13}\}$.

2. Fill in the blanks in statements (A) through (D) with the appropriate phrase [1], [2], [3] or [4] listed below:

(A) Given sets A, B, $A \cup B = A$ is $\{\underline{14}\}$ for $A \cap B = B$.

(B) For some integer n, n^2 being some multiple of 12 is $\{\underline{15}\}$ for n being a multiple of 12.

(C) The center of the circle inscribed in triangle T coinciding with the center of the circle which circumscribes triangle T is $\{\underline{16}\}$ for triangle T to be an equilateral triangle.

(D) Given real numbers a, b, and c,

$$|a + b + c| = |a| + |b| + |c|$$

is $\{\underline{17}\}$ for $ab + bc + ca \geq 0$.

[1] a necessary and sufficient condition

[2] a necessary but not sufficient condition

[3] a sufficient but not necessary condition

[4] neither a sufficient nor a necessary condition

Section 2 (35 points)

Let a be a constant. Consider the parabola

$$C_a : y = -x^2 + ax + a^2.$$

1. Since the coordinates of the vertex of C_a are

$$\left(\frac{a}{\{\underline{1}\}}, \frac{\{\underline{2}\}}{\{\underline{3}\}}a^2\right),$$

the vertex is on the curve $y = \{\underline{4}\}x^2$.

2. Let ℓ be the line joining two points $A\ (-1, 1)$ and $B\ (2, 4)$. For the parabola C_a and the line ℓ to have a common point, the value of a must be

$$a \leq \{\underline{5}\}\{\underline{6}\} \qquad \text{or} \qquad a \geq \frac{\{\underline{7}\}}{\{\underline{8}\}}.$$

The coordinates of the common point of parabola C_a and the line ℓ are

$$(\{\underline{9}\}\{\underline{10}\}, \{\underline{11}\}), \qquad \text{when } a = \{\underline{5}\}\{\underline{6}\}$$

and

$$\left(\frac{\{\underline{12}\}}{\{\underline{13}\}}, \frac{\{\underline{14}\}}{\{\underline{15}\}}\right), \qquad \text{when } a = \frac{\{\underline{7}\}}{\{\underline{8}\}}.$$

Also, in order for the parabola C_a and the line segment AB to have two distinct points of intersection,

$$\frac{\{\underline{16}\}}{\{\underline{17}\}} < a \leq \{\underline{18}\}.$$

Section 3 (35 points)

Consider the triangle ABC with coordinates $A\ (0, 3)$, $B\ (-1, 0)$ and $C\ (2, 1)$.

1. The center of the circumscribed circle is

$$\left(\frac{\{\underline{1}\}}{\{\underline{2}\}}, \frac{\{\underline{3}\}}{\{\underline{4}\}}\right),$$

and the radius is

$$\frac{\{\underline{5}\}\sqrt{\{\underline{6}\}}}{\{\underline{7}\}}.$$

Also,

$$\sin\angle ABC = \frac{\{\underline{8}\}}{\{\underline{9}\}},$$

and the area of triangle ABC is $\{\underline{10}\}$.

From these conditions, we know that the radius of the inscribed circle is

$$\frac{\sqrt{\{\underline{11}\}\{\underline{12}\}} - \sqrt{\{\underline{13}\}}}{\{\underline{14}\}}.$$

2. If point P moves along the sides of triangle ABC, the maximum value of the distance between the origin O and point P is $\{\underline{15}\}$ and the minimum value is

$$\frac{1}{\sqrt{\{\underline{16}\}\{\underline{17}\}}}.$$

Answers to Mathematics A [Mathematics I] (100 points)

Section 1

1.	$a = \{\underline{1}\}$	$a = 2$
	$\{\underline{2}\}b + c = \{\underline{3}\}\{\underline{4}\}\{\underline{5}\}$	$3b + c = 127$
	$\{\underline{6}\}\{\underline{7}\}b + c = \{\underline{8}\}\{\underline{9}\}$	$-7b + c = 37$
	$b = \{\underline{10}\}$	$b = 9$
	$c = \{\underline{11}\}\{\underline{12}\}\{\underline{13}\}$	$c = 100$
2.	$\{\underline{14}\}$	1
	$\{\underline{15}\}$	2
	$\{\underline{16}\}$	1
	$\{\underline{17}\}$	3

Section 2

1.	$\left(\frac{a}{\{\underline{1}\}}, \frac{\{\underline{2}\}}{\{\underline{3}\}}a^2\right)$	$\left(\frac{a}{2}, \frac{5}{4}a^2\right)$
	$\{\underline{4}\}x^2$	$5x^2$
2.	$a \leq \{\underline{5}\}\{\underline{6}\}$	$a \leq -1$
	$a \geq \{\underline{7}\}/\{\underline{8}\}$	$a \geq 7/5$
	$(\{\underline{9}\}\{\underline{10}\}, \{\underline{11}\})$	$(-1, 1)$
	$\left(\frac{\{\underline{12}\}}{\{\underline{13}\}}, \frac{\{\underline{14}\}}{\{\underline{15}\}}\right)$	$\left(\frac{1}{5}, \frac{11}{5}\right)$
	$\{\underline{16}\}/\{\underline{17}\} < a \leq \{\underline{18}\}$	$7/5 < a \leq 2$

Section 3

1.	$\left(\frac{\{\underline{1}\}}{\{\underline{2}\}}, \frac{\{\underline{3}\}}{\{\underline{4}\}}\right)$	$\left(\frac{1}{4}, \frac{5}{4}\right)$
	$\frac{\{\underline{5}\}\sqrt{\{\underline{6}\}}}{\{\underline{7}\}}$	$\frac{5\sqrt{2}}{4}$
	$\frac{\{\underline{8}\}}{\{\underline{9}\}}$	$\frac{4}{5}$
	$\{\underline{10}\}$	4
	$\frac{\sqrt{\{\underline{11}\}\{\underline{12}\}} - \sqrt{\{\underline{13}\}}}{\{\underline{14}\}}$	$\frac{\sqrt{10} - \sqrt{2}}{2}$
2.	$\{\underline{15}\}$	3
	$\frac{1}{\sqrt{\{\underline{16}\}\{\underline{17}\}}}$	$\frac{1}{\sqrt{10}}$

Mathematics B [Mathematics II] (100 points, 60 minutes)

Choose two of the following three sections:

Section 1 (50 points)

1. In a circle with radius 2 and its center at the origin O, let the vertices of an inscribed hexagon be $ABCDEF$, with the coordinates of A at $(2, 0)$, and with B in the first quadrant.

(1) The components of the vector

$$\overrightarrow{AB} + 2\overrightarrow{DE} - 3\overrightarrow{FA}$$

are $\left(\{\underline{1}\}\{\underline{2}\}, \{\underline{3}\}\{\underline{4}\}\sqrt{\{\underline{5}\}}\right)$.

(2) If t is a real number, the magnitude of the vector

$$\overrightarrow{AB} + t\overrightarrow{EF}$$

is a minimum when the value of t is

$$\frac{\{\underline{6}\}}{\{\underline{7}\}}$$

and the minimum magnitude is $\sqrt{\{\underline{8}\}}$.

2. Let $ABCD$ be a quadrilateral with

$$\overrightarrow{BC} = 2\overrightarrow{AD}$$

$$AB = CD = DA = 2$$

$$\overrightarrow{AD} = \overrightarrow{a}$$

$$\overrightarrow{BA} = \overrightarrow{b}$$

(1) Let M be the midpoint of CD. Since $\angle BCM = \{\underline{9}\}\{\underline{10}\}^\circ$;

$$BM = \sqrt{\{\underline{11}\}\{\underline{12}\}}.$$

Also

$$\overrightarrow{BM} = \frac{\{\underline{13}\}}{\{\underline{14}\}}\overrightarrow{a} + \frac{\{\underline{15}\}}{\{\underline{16}\}}\overrightarrow{b}. \qquad (1)$$

(2) Let P be a point on AB, and let Q be the point of intersection of PC and BM. Suppose $PQ : QC = 1 : 2$. Let us find $AP : PB$ and $BQ : QM$. If we set

$$\overrightarrow{BP} = t\overrightarrow{BA},$$

we have

$$\overrightarrow{BQ} = \frac{\{\underline{17}\}}{\{\underline{18}\}}(\overrightarrow{a} + t\,\overrightarrow{b}). \qquad (2)$$

Therefore, from (1) and (2),

$$t = \frac{\{\underline{19}\}}{\{\underline{20}\}}.$$

It follows that $AP : PB = \{\underline{21}\} : \{\underline{22}\}$, $BQ : QM = \{\underline{23}\} : \{\underline{24}\}$ and

$$BQ = \frac{\{\underline{25}\}}{\{\underline{26}\}}\sqrt{\{\underline{27}\}\{\underline{28}\}}.$$

Section 2 (50 points)

1. The function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-(2\sqrt{3})/9$ at $x = 1/\sqrt{3}$. Then,

(1) $a = \{\underline{1}\}$, $b = \{\underline{2}\}\{\underline{3}\}$ and the local maximum value of the function $f(x)$ is

$$\frac{\{\underline{4}\}\sqrt{\{\underline{5}\}}}{\{\underline{6}\}}.$$

(2) The value of the slope m of the tangent line at point P on the curve $y = f(x)$ is greater than or equal to $\{\underline{7}\}\{\underline{8}\}$. If $m = \tan\theta$ $(0^\circ \le \theta < 180^\circ)$, then the range of values for θ is $0^\circ \le \theta < \{\underline{9}\}\{\underline{10}\}^\circ$ or $\{\underline{11}\}\{\underline{12}\}\{\underline{13}\}^\circ \le \theta < 180^\circ$.

(3) The volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = f(x)$ about the x-axis is

$$\frac{\{\underline{14}\}\{\underline{15}\}}{\{\underline{16}\}\{\underline{17}\}\{\underline{18}\}}\pi.$$

2. $1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \ldots$ is a sequence where $1/2^{k-1}$ appears 2^k times successively $(k = 1, 2, 3, \ldots)$.

(1) Then the sum of the first 1000 terms is

$$\{\underline{19}\} + \frac{\{\underline{20}\}\{\underline{21}\}\{\underline{22}\}}{2^{\{\underline{23}\}}}.$$

(2) If the sum of the first n terms is 100, then because

$$n = 2^{\{\underline{24}\}\{\underline{25}\}\{\underline{26}\}} - \{\underline{27}\},$$

n is a $\{\underline{28}\}\{\underline{29}\}$ digit number provided that $\log_{10} 2 = 0.3010$.

Section 3 (50 points)

The numbers 1 through 9 are written individually on nine cards. Choose three cards from the nine, letting x, y, and z denote the numbers of the cards arranged in increasing order.

1. There are $\{\underline{1}\}\{\underline{2}\}$ such x, y, and z combinations.

2. The probability of having x, y, and z all even is

$$\frac{\{\underline{3}\}}{\{\underline{4}\}\{\underline{5}\}}.$$

3. The probability of having x, y, and z be consecutive numbers is

$$\frac{\{\underline{6}\}}{\{\underline{7}\}\{\underline{8}\}}.$$

4. The probability of having $x = 4$ is

$$\frac{\{\underline{9}\}}{\{\underline{10}\}\{\underline{11}\}}.$$

5. Possible values of x range from $\{\underline{12}\}$ to $\{\underline{13}\}$. If k is an integer such that $\{\underline{12}\} \le k \le \{\underline{13}\}$, the probability of $x = k$ is

$$\frac{(\{\underline{14}\} - k)(\{\underline{15}\} - k)}{\{\underline{16}\}\{\underline{17}\}\{\underline{18}\}}.$$

The expected value of x is

$$\frac{\{\underline{19}\}}{\{\underline{20}\}}.$$

Answers to Mathematics B [Mathematics II] (100 points)

Section 1

1.	$(\{\underline{1}\}\{\underline{2}\}, \{\underline{3}\}\{\underline{4}\}\sqrt{\{\underline{5}\}})$	$(-4, -4\sqrt{3})$
	$\{\underline{6}\}/\{\underline{7}\}$	$1/2$
	$\sqrt{\{\underline{8}\}}$	$\sqrt{3}$
2.	$\{\underline{9}\}\{\underline{10}\}$	60
	$\sqrt{\{\underline{11}\}\{\underline{12}\}}$	$\sqrt{13}$
	$\frac{\{\underline{13}\}}{\{\underline{14}\}}\vec{a} + \frac{\{\underline{15}\}}{\{\underline{16}\}}\vec{b}$	$\frac{3}{2}\vec{a} + \frac{1}{2}\vec{b}$
3.	$\{\underline{17}\}/\{\underline{18}\}$	$2/3$
	$\{\underline{19}\}/\{\underline{20}\}$	$1/3$
	$\{\underline{21}\} : \{\underline{22}\}$	$2 : 1$
	$\{\underline{23}\} : \{\underline{24}\}$	$4 : 5$
	$\frac{\{\underline{25}\}}{\{\underline{26}\}}\sqrt{\{\underline{27}\}\{\underline{28}\}}$	$\frac{4}{9}\sqrt{13}$

Section 2

1.	$\{\underline{1}\}$	0
	$\{\underline{2}\}\{\underline{3}\}$	-1
	$\frac{\{\underline{4}\}\sqrt{\{\underline{5}\}}}{\{\underline{6}\}}$	$\frac{2\sqrt{3}}{9}$
	$\{\underline{7}\}\{\underline{8}\}$	-1
	$\{\underline{9}\}\{\underline{10}\}$	90
	$\{\underline{11}\}\{\underline{12}\}\{\underline{13}\}$	135
	$\frac{\{\underline{14}\}\{\underline{15}\}}{\{\underline{16}\}\{\underline{17}\}\{\underline{18}\}}$	$\frac{16}{105}$
2.	$\{\underline{19}\}$	9
	$\frac{\{\underline{20}\}\{\underline{21}\}\{\underline{22}\}}{2^{\{\underline{23}\}}}$	$\frac{489}{2^9}$
	$n = 2^{\{\underline{24}\}\{\underline{25}\}\{\underline{26}\}} - \{\underline{27}\}$	$n = 2^{100} - 1$
	$\{\underline{28}\}\{\underline{29}\}$	31

Section 3

1.	$\{\underline{1}\}\{\underline{2}\}$	84
2.	$\frac{\{\underline{3}\}}{\{\underline{4}\}\{\underline{5}\}}$	$\frac{1}{21}$
3.	$\frac{\{\underline{6}\}}{\{\underline{7}\}\{\underline{8}\}}$	$\frac{1}{12}$
4.	$\frac{\{\underline{9}\}}{\{\underline{10}\}\{\underline{11}\}}$	$\frac{5}{42}$
5.	$\{\underline{12}\}$, $\{\underline{13}\}$	1, 7
	$\frac{(\{\underline{14}\} - k)(\{\underline{15}\} - k)}{\{\underline{16}\}\{\underline{17}\}\{\underline{18}\}}$	$\frac{(9-k)(8-k)}{168}$ or $\frac{(8-k)(9-k)}{168}$
	$\frac{\{\underline{19}\}}{\{\underline{20}\}}$	$\frac{5}{2}$

Results of Performance on 1990 UECE

EXAMINATION	Number of Participants	Average	High	Low	Standard Deviation
MATHEMATICS A					
Mathematics I	353,010	73.37	100	0	23.43
MATHEMATICS B					
Mathematics II	327,034	64.27	100	0	22.62
Industrial Math.	52	40.87	91	0	23.69
Accounting/Stat. I, II	457	62.42	99	11	17.84

Evaluation of the 1990 University Entrance Center Examination
(Direct translation of the text)

I. Opinions of and Evaluations by Senior High School Teachers

1. Preface

Discussions and research into establishing a more suitable university entrance examination to accommodate a great increase in the number and diversity of university applicants have allowed private universities to participate along with the national universities.

Accordingly, we revised the contents of the examinations. Originally, Math I and Math II combined were 100 minutes, 200 points; the UECE in mathematics is currently divided into two groups: Math A [Math I] and Math B [Math II], each exam being 60 minutes long and worth 100 points.

The purpose of the UECE is, as before, to assess the degree of mastery of the general fundamental learning established for the senior high schools. Considering the various ways the UECE is utilized in selecting university entrants, fairness to the exam participants was emphasized by posing the following questions:

(1) whether exam problems (content, questions, score distributions, format, etc.) accurately assess the level of fundamental learning;
(2) whether there is a wide difference in the degree of difficulty between the main exam and the make-up exam;
(3) whether there is a great difference in the degree of difficulty among optional problem sections.

With these criteria in mind, we will analyze the 1990 exam problems from the following points of view:

(1) whether the exam problems fulfill the purpose of the UECE exam;
(2) whether Math I, Math II are consistent with the guidelines set forth by the senior high schools;
(3) whether the content of exam problems tends to be esoteric;
(4) whether the degree of difficulty of optional problem sections for Math II is uniform in the three sections;
(5) whether the problems provide sufficient information to evaluate the student's mathematical thinking ability and ability to perform calculations;
(6) whether, for each problem, the level of difficulty, the method of questioning, the distribution of points, format, and so forth are appropriate;
(7) whether the problems reflect an improvement resulting from reviewing the past JFSAT and the Trial Center Exam administered in 1988.

2. Content and Scope of the Exam Problems

Mathematics I

Section 1. Numbers and expressions, equations and inequalities

1. Polynomials with integer coefficients and equations

Using the two conditions given and the remainder theorem, divide the integral expression by the second-degree expression.

2. Proofs and statements

Covering combinations, integers, graphs and inequalities from Math I, determine whether the condition given in each part is necessary, sufficient, or both.

Section 2. Functions, two-dimensional graphs and expressions

1. Find the coordinates of the vertex of a parabola and the equation of its locus.

2. Find the condition for having common points between the line joining the given two points and the parabola, and find the coordinates of the common points. Also, find the condition that the line segment AB and the parabola have two distinct common points.

Section 3. Two-dimensional graphs and related expressions, trigonometric ratios

1. Find the center and the radius of the circle passing through three points A, B, and C in a plane. Also, find $\sin\angle ABC$ and the area of $\triangle ABC$ using either the law of sines or the law of cosines. Then find the radius of the inscribed circle.

2. Find the maximum and minimum distances between the origin and a point on a side of $\triangle ABC$.

Mathematics II

Section 1. Vectors

1. Coordinatized two-dimensional vectors

(1) Determine the coordinates of the given points and then find the components of the vector.
(2) Find the minimum value of the magnitude of the vector.

2. Two-dimensional vectors

(1) Using the conditions given for the vectors, draw the graph and find the angle. Using the law of cosines, find the length of a side. Also, express the vector in terms of two linearly independent vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Section 2. Progressions, derivatives, integrals, and trigonometric functions

1. Derivatives, integrals, and trigonometric functions

(1) Determine the coefficients from the information given about the minimum value of the function. Also determine the maximum value of the function.
(2) Make use of the derivative to find the minimum value for the slope. Find the range of the angle formed by the tangent and the positive x-axis.
(3) Find the volume of the solid generated by revolving the region bounded by the curve and the x-axis about the x-axis.

2. Progressions and logarithms

(1) Consider the pattern of the progression, and make use of the sum formula for the geometric progression. This problem is beyond the curriculum guidelines established for the senior high schools and requires special effort.
(2) Given the sum, use the geometric progression to determine the number of terms n. Also, use the common logarithm to find the digits of n.

3. Probability

(1) Determine the value of the combination.
(2) Determine the probability of choosing three numbers from $\{2, 4, 6, 8\}$.
(3) Determine the probability of choosing three consecutive numbers.
(4) Determine the probability of choosing three numbers, of which 4 is the smallest number.
(5) If the smallest of three numbers is k, find the range for k. Also find the probability of k being the smallest number and then find the expected value of the smallest number.

3. Analysis of the Exam Problems

Mathematics I

Section 1

1. This basic problem using the remainder theorem is typical for a university entrance exam. If one knows the remainder theorem, it is unthinkable that one should miss this problem, aside from errors in calculations. This problem is suitable for testing the level of the basic learning established for senior high schools.

2. The format for this problem on expressions and proofs is multiple-choice because it was assumed that more participants could answer the problems intuitively.

(1) In set theory, $(A \cup B) \supset B$ and $A \supset (A \cap B)$ are basic knowledge, but the senior high school textbooks do not cover this in detail. We think that many answered hastily without considering the proof.

(2) If n is a multiple of 12, then n^2 is a multiple of 12. This is obvious. To disprove the converse is not too difficult.

(3) If T is an equilateral triangle, then the center of the inscribed circle and that of the circumscribed circle coincide. This can be proven with the knowledge gained from junior high school mathematics. We think many chose the answer [A] without considering the proof.

(4) Perhaps it did not occur to the students to square both sides of

$$|a+b+c| = |a| + |b| + |c|$$

and even if it had occurred to them, it would have appeared that finding a counterexample to the converse required too much time and work. We feel that more students considered the problem intuitively.

In these types of multiple-choice questions, it is difficult to see the logical thinking process. We would like to see an improvement made in the problem format; one which emphasizes the thought process that leads to the conclusion, rather than a format that encourages guessing the answer to the problem. We assume that this type of theorem-proof problem develops logical thinking.

Section 2

1. This is a basic problem. We think that the short questions in the first part were easy and not confusing. The coordinates of the vertex can be obtained from the standard second-degree form. The equation of the locus can be found easily by eliminating the parameter a. This problem expresses the parabola by C_a, a notation rarely used in textbooks.

2. Parts $\{\underline{1}\}$ through $\{\underline{15}\}$ are textbook-level basic questions which are considered easy to solve. For the remaining parts, it is important to keep in mind that

$$x^2 - (a-1)x - a^2 + 2 = 0 \quad \text{and} \quad -1 \le x \le 2$$

in order to have two distinct real solutions. This is a good question to test graphing ability and mathematical thinking. We feel participants were unfamiliar with this type of problem; they either thought that they did not have enough conditions or miscalculated somewhere. Overall, the number of questions and the distribution of points are well-balanced. These are good standard questions.

Section 3

1. In order to find the center of the circumscribed circle of $\triangle ABC$, solve a system of simultaneous equations by setting the equation of the circumscribed circle to be

$$x^2 + y^2 + ax + by + c = 0,$$

or solve for the intersecting point of two perpendicular bisectors of the sides. We suspect that there were students who did not follow the proper order in these questions and who used $\sin \angle ABC$ in the next part to solve for the radius of the circumscribed circle. The radius of the inscribed circle can be obtained by the formula

$$S = rs.$$

Depending on the textbook, the exact specification of this formula varies.

2. The shortest distance between the origin O and the point P can be found simply by using the formula for the distance between a point and a line. Since the forms of the formula vary depending on the textbook, this problem requires some work. Overall, it is a good question. There is no objection to its level, distribution of points, or format.

Mathematics II

Section 1

1. (1) The positions of the vertices of the regular hexagon can be obtained by using the trigonometric ratios or symmetric points. We think it might have been better to ask the participants to find the coordinates of point B or point F before asking (1).

(2) Expressing the components of the vector

$$\overrightarrow{AB} + t\overrightarrow{EF},$$

the minimum value of the magnitude of this vector can be found from a second-degree function in t. Because there is another way to solve the problem using the inner product, the problem should be designed to ensure fairness to all participants.

2. (1) BM can be obtained by the law of cosines or the Pythagorean theorem. Considering

$$\overrightarrow{BM}$$

as the midpoint vector of CD, BM can be found easily. Students easily solved this problem.

(2) From

$$\overrightarrow{BQ} = 1/3\left(2\overrightarrow{BP} + \overrightarrow{BC}\right),$$

one can find

$$\frac{\{\underline{17}\}}{\{\underline{18}\}}.$$

Setting

$$\overrightarrow{BM} = k\overrightarrow{BQ},$$

from the fact that $\overrightarrow{a}$ and $\overrightarrow{b}$ are linearly independent, one can find the value of t. From this fact, $\{\underline{21}\}$, $\{\underline{22}\}$, $\{\underline{23}\}$, and $\{\underline{24}\}$ can be found. In particular, when making use of equation (1) to find BQ, we think some students gave up in the middle of the process due to lack of graphic observation. Although it was stated in the problem that

$$\overrightarrow{BA} = \overrightarrow{b},$$

it might have been clearer if it had been specified that

$$\overrightarrow{AB} = \overrightarrow{b} \quad \text{because} \quad \overrightarrow{AD} = \overrightarrow{a}.$$

Section 1 is a set of good questions which tie the graphs to the vectors. In terms of the level of difficulty, distribution of points, and format, they are well-constructed questions which determine whether basic concepts are well understood.

Section 2

1. (1) All are basic problems.

(2) We can see that an effort was made to demonstrate the steps in a general problem dealing with differentiation and trigonometric functions.

(3) Here is a basic problem with calculations that are not very complex. Overall, (3) is a good question. However, if (1) is not solved, then (2) and (3) cannot be solved, and the motive for asking (2) and (3) is lost. We hope to see a modification in the problem format.

2. (1) The steps for these problems were very difficult for the students. We would like to have first asked a more concrete question such as "find the sum of the first 10 terms" to demonstrate the pattern. Eight points are assigned for answers $\{\underline{20}\}$ through $\{\underline{23}\}$. In order to give credit for steps in the solving process, we would have preferred to assign 4 points to the numerator and 4 points to the denominator.

(2) If one observes that there are 2^{k-1} terms of $1/(2^{k-1})$ and that these terms sum to 1, the problem can be easily solved; however, it was probably a very hard question for those who were unaccustomed to this concept. Also, regarding the problems on the number of the digits, if one had not observed that

$$\log(2^{100} - 1) \approx \log 2^{100},$$

or if one were not accustomed to dealing with logarithms, we think this problem would be difficult to solve. Part 2 is a good application problem involving progressions and testing mathematical thinking ability; however, we think the problem is beyond the scope of textbooks, and we would have liked to have seen a concrete example leading step-by-step to the solution.

Section 3

1. Basic problem. Because of the nature of probability, the answer to (1) is related to all parts in (2) and all parts thereafter. It is an appropriate question.

2. Basic problem.

3. If one observes that once x is determined, y, z are determined, then it is a simple and basic problem.

4. Selecting y and z is the same as finding the combination of selecting 2 cards from 5 cards which are marked 5, 6, 7, 8, and 9. This is also a basic problem.

5. From $\{\underline{12}\}$, $\{\underline{13}\}$, one can solve the problem if one understands the point of the question. To find the probability of $x = k$, one can generalize the method for 4. Since the students might not have been accustomed to calculating combinations involving a letter (rather than a number), we consider this a bad problem.

Section 3 is a set of good questions overall.

4. Summary

A summary of those points we felt strongly about on this year's Center Examination and a record of our suggestions follows:

1. This year's averages are 73 for Math I and 64 for Math II, higher than the target scores of 60%. It has won much approval from senior high school teachers. Overall, there are many good, standard problems with well-researched content. The problems were appropriate for testing the degree of achievement in the senior high schools.

2. The content of Math I was sufficiently consistent with the content in the learning guidelines for the senior high schools. In Math II, the content of Sections 1 and 2 were adequate, but the level of Section 2.2 on grouped progressions was beyond the scope of the textbooks used. However, this type of application problem is useful in determining the student's innovative mathematical thinking ability. It is hoped that an improved form of the question can be developed.

3. The problem on progressions crosses the entire range of Math I and Math II. There were many good questions testing the student's ability to integrate their mathematical knowledge. We do not consider the content obscure, but rather well-balanced.

4. The optional problem sections in Math II were all expected to have been of the same level of difficulty, but Section 2.2 included a progression which students were not used to, making the problem difficult. There is considerable disparity between the levels of difficulty in Sections 1 and 2. For a higher level problem like 2.2, we would have liked to have seen more attention devoted to the questioning method and to suggest such approaches as giving a concrete example and asking the question in parts.

5. The 1990 exam problems are consistently appropriate for evaluating basic mathematical thinking and calculation ability. We see the intent of the UECE as testing the student's mastery of essential mathematical material. For Math I, Section 1.2, we would like to see an additional true-false section, more testing of the problem-solving thought process, and the elimination of ambiguous questions.

6. For JFSAT, the exam time for both Math I and Math II combined was 100 minutes. For UECE, though the time allotted for simple problems was transferred to harder problems within each exam, it seemed that 60 minutes each for Math I and Math II was not sufficient. In order to determine more accurately the student's mathematical ability, we wonder if it is possible to cover the same material and shorten the rest period in order to lengthen each testing period to 80 minutes.

7. As for topics covered, the exam has been improved by the inclusion of problems on progressions, reflecting efforts to respond to earlier criticisms and suggestions. To devise fair, valid problems is a difficult task, but we hope that efforts toward this goal will be continued.

II. Analysis of 1990 UECE by Division of Research, Senior High School Division, Association of Japanese Mathematical Education

1. Guidelines for Exam Problems

The University Entrance Center Examination (UECE) was given for the first time this year. The previous JFSAT [Math I, Math II] was divided into Math A [Math I] and Math B [Math II], each exam being 60 minutes long and worth 100 points. The basic guidelines of the exam problems have not been changed; the UECE is designed to measure the student's level of mathematical achievement in the senior high school. The exam is to be used by individual universities to select entrants.

In order to evaluate the student's achievement in basic mathematics, we need to avoid ambiguous or misleading problems. The exam problems should be such that students could answer them solely on the basis of their understanding of the textbooks and their ability to apply that understanding; the scores should accurately reflect the level of understanding of the problems. Since this exam is one criterion used for judging a university entrant, we must, of course, avoid problems which all students can solve and problems which none can solve. Particularly for those universities which do not require math in the secondary examination, we need to ensure that exam problems provide sufficient information for the selection of the entrants. Also, the degree of the difficulty of the optional problem sections in Math II should not be widely disparate.

2. On This Year's Problems

1. In keeping with the above mentioned guidelines, we devised problems of the appropriate level by taking the following into account:

i. the results of the analysis of the previous JFSAT;
ii. the report of the 1989 exam committee regarding points needing to be addressed in future exams;
iii. opinions and criticisms of senior high school teachers and the Association of Japanese Mathematical Education.

As for the degree of difficulty, we aimed to maintain an average score of around 60 points.

2. Math I has three sections that were required of all students; in Math II the student has a choice of two of the three sections. We made sure that we gave problems in diverse areas.

3. Students' Group Divisions and Performance

For this year's exam, there were 353,010 students who took Math A with an average score of 73.37. There were 327,543 students who took Math B; of those, 52 took engineering math and 457 took business mathematics. 327,034 students took Math II, with an average score of 64.27. For each of the three sections in Math II, the number of participants who selected the problem and the average scores are shown in the following table:

Section	1	2	3
Number of Participants	301,190	263,030	89,832
Average (50 Points)	36.11	27.96	31.03

4. Content and Intent of Math I, Math II Exam Problems, and Exam Results

The content of the problems was as follows:

Mathematics I

Section 1. Second-degree equations, necessary and sufficient conditions

Section 2. Parabolas

Section 3. Trigonometric ratios

Mathematics II

Section 1. Vectors

Section 2. Differentiation, integration, sequences [progressions]

Section 3. Probability

The intent of each problem was as follows:

Mathematics I

Section 1

1. Determine the student's understanding regarding the quotient and the remainder of second-degree polynomials with integer coefficients.

2. Determine the degree of understanding regarding the logic of necessary and sufficient conditions.

Section 2

On the topic of the relationship between a parabola and a line, determine knowledge of the basic facts regarding second-degree functions and their graphs.

Section 3

On the topic of inscribed and circumscribed circles of a triangle, determine knowledge of basic facts regarding trigonometric ratios.

Mathematics II

Section 1

1. Determine the student's understanding of and ability to apply the basic facts regarding two-dimensional vectors.

Section 2

1. Determine the student's understanding of the basic facts related to differentiation and integration of a third-degree polynomial.

2. Test the student's thinking ability and understanding of geometric progressions using a sequence of fractions.

3. Test the student's ability to calculate the probability of a combination of given numbers.

The following indicates the percentage of correct answers, incorrect or incomplete answers, and no answers for each problem.

Mathematics I

Section 1

1. One would know how to solve this basic remainder theorem problem by factoring three second-degree expressions. The percentage of correct answers for each part was high, above 80%.

2. This problem tests the theoretical thought process involving necessary and sufficient conditions. $\{\underline{16}\}$ can be solved by looking at the graph. The percentage of correct answers was high: 81.38%. However, the percentage of correct answers on other parts were unexpectedly low (57.4% for $\{\underline{14}\}$, 60.48% for $\{\underline{15}\}$, 49.22% for $\{\underline{17}\}$); perhaps students were not used to finding counterexamples.

Section 2

This section tests basic understanding and application ability related to second-degree curves. The percentage of correct answers for Part 1 was 85%. Though overall performance was good, we think the students were not used to eliminating parameters. Part 2 investigates the situation when a parabola and a line intersect. With the exception of the last inequality, the percentage of correct answers was above 85%, an impressive performance. The average for Section 2 was 27.84 (out of 35 points), the best among the three sections.

Section 3

This is a standard problem involving two-dimensional graphs, expressions, and trigonometric ratios. The scoring rate was about 67%. In Part 1 the problem is to find the center and the radius of $\triangle ABC$'s circumscribed circle and to find the radius of the inscribed circle. A surprisingly low percentage, 59%, of the students were able to find the center of the circumscribed circle.

We think that the great difference between the percentage of correct answers for finding the area of $\triangle ABC$ (76%) and the percentage of correct answers for finding the radius of the inscribed circle (about 55%) was because some participants did not know the formula $S = rs$. Part 2 asked for the maximum and minimum values of the distance between the origin and a point P moving along the edges of the $\triangle ABC$. When the question was given, we were concerned with whether or not the students knew the formula for the distance between a point and the line, but the results indicate that the difficulty level of this question was not a problem.

Mathematics II

Section 1

This section tests the basics and applications of vectors using two-dimensional graphs. The selection rate was 92%, and the average was 36.11.

1. Calculate a position vector and its magnitude. The percentage of correct answers was high, about 75%.

2. This part tests the application of the law of cosines for triangles and the ability to find the position vector of an interior point and the ratio of the magnitudes of vectors. The percentage of correct answers moved downward from 90% to 45% in this part.

Section 2

1. (1) Find the function from its minimum value and then find its maximum value.
(2) Bound the slope of the tangent, and find the range of the angle formed by the tangent and the positive x-axis.
(3) Find the volume of a solid of revolution. It is a standard integration problem. The scoring rate of Part 1 was 75%.

2. (1) This problem is to note the pattern of the sequence. It is useful to express the number of terms using a geometric progression and note that $2^9 = 512 < 1000 < 2^{10} = 1024$.
(2) This problem asks one to consider the converse. It was devised to test the student's thinking and application abilities. The percentage of correct answers was unexpectedly low (26% for the first part and 14.7% for the second part) and many left this problem blank.

Section 3

Find the probability of picking three cards out of 9 cards. Parts 1, 2, 3, and 4 were basic, and the percentage of correct answers ranged from 63% to 87%. Well done. Because Part 5 used the letter k, there were many wrong answers and blank answers; the percentage of correct answers was cut down to 30%. The overall average of Section 3 was 31.03 (out of 50 points possible), which is considered high for a probability problem.

5. Analysis of Criticism and Opinions on Exam Problems

(1) Overall this year's exam problems were found to be basic, and suitable to test the achievement level of learning steps in the senior high schools.
(2) On this year's Math II, the problem on sequences in Section 2, Part 2 was good; others said it was beyond the scope of the textbooks of the senior high schools. It is regrettable but, because of the limit on the number of blanks, we could not set the question up in steps.
(3) In the make-up exam for Math II, Section 1, one of the angles was expressed in radians. According to the guidelines of the UECE, it should have been in degrees. The students were not confused, but we should be more cautious in constructing problems in the future.

6. The Make-up Exam

The number of participants in this year's make-up exam was 201 for Math I and 191 for Math II. The content of Math I problems was as follows:

Section 1. The least common multiple and the largest common factor of two third-degree polynomials with integer coefficients;

Section 2. Parabolas;

Section 3. Trigonometric ratios and the circle on the two-dimensional coordinate plane.

The content of Math II problems was as follows:

Section 1. Vectors;

Section 2. Sequences, trigonometric functions, third-degree functions;

Section 3. Probability.

Examinees were to choose two sections from the above three. After adjusting the degree of difficulty of the original exam, this year's make-up exam was considered adequate on the whole; however, it seemed that the problem on probability was somewhat difficult.

7. Points to be Considered in Constructing Problems in the Future

(1) We should take care not to go beyond the range of the guidelines for the senior high school studies. For those problems requiring a high level of thinking, we should consider whether or not questions should be asked in stages.

(2) Regarding the necessary-sufficient condition problem or any theorem-proof type problem, some judged that it could not be expected to test thinking ability. We should put more effort toward avoiding those problems which many students can answer without understanding the content.

(3) Following the basic guidelines stated in 1 and being careful to avoid wide variations in the degree of difficulty in the optional problem sections, we would like to maintain the 60% average score in the future.

Survey on University Entrance Center Examination*

(Division of Research, Senior High Schools Division Association, Association of Japanese Mathematical Education)

Please circle the suitable choices among [1] through [21] in the following and enter the appropriate number in the blank for question 6.

Your field of study:

[1] science, engineering, medicine, pharmacology, agriculture
[2] other

1. Your exam subject:

[3] Math I only
[4] Math I and Math II

2. Compared to the section and chapter review problems of the textbooks, overall, the exam problems on Math I were

[5] easier
[6] about the same level
[7] harder

and on Math II were

[8] easier
[9] about the same level
[10] harder

3. Level of the problems:

[11] it was sufficient to study the important points of the textbooks
[12] it was necessary to prepare for the exam

4. Number of problems on Math I:

[13] too few
[14] just right
[15] too many

Number of problems on Math II:

[16] too few
[17] just right
[18] too many

5. Problems you solved in Math II:

[19] [Algebra, Geometry] and [Basic Analysis]
[20] [Algebra, Geometry] and [Probability, Statistics]
[21] [Basic Analysis] and [Probability, Statistics]

6. Which problem in Math I did you find the most difficult? Problem number ____ .

* The 1990 UECE was held on January 13, 14 (the make-up exam on January 20, 21). This survey was made on February 8, 1990 among those seniors who were expecting to graduate in March 1990 from 31 national and local public high schools in the Tokyo metropolitan area.

RESULTS OF SURVEY ON UNIVERSITY ENTRANCE CENTER EXAMINATION

February 8, 1990

	SCIENCE MAJORS		HUMANITIES MAJORS		SCIENCE & HUMANITIES MAJORS	
Response	Number of Responses	Percentage of Responses	Number of Responses	Percentage of Responses	Number of Responses	Percentage of Responses
[3]	42	4.34	144	20.90	186	11.23
[4]	926	95.66	545	79.10	1,471	88.77
[5]	293	30.49	166	23.65	459	27.60
[6]	478	49.74	375	53.42	853	51.29
[7]	190	19.77	161	22.93	351	21.11
[8]	178	19.45	77	13.73	255	17.28
[9]	449	49.07	266	47.42	715	48.44
[10]	288	31.48	218	38.86	506	34.28
[11]	512	54.07	350	51.09	862	52.82
[12]	435	45.93	335	48.91	770	47.18
[13]	30	3.11	13	1.92	43	2.62
[14]	732	75.93	497	73.52	1,229	74.94
[15]	202	20.95	166	24.56	368	22.44
[16]	33	3.60	22	3.87	55	3.71
[17]	622	67.90	363	63.91	985	66.37
[18]	261	28.49	183	32.22	444	29.92
[19]	570	61.36	324	57.65	894	59.96
[20]	207	22.28	167	29.72	374	25.08
[21]	152	16.36	71	12.63	223	14.96

Survey sample of 31 schools.

Conclusion

The translations of the evaluation of the UECE and the analysis of the exam parts in the previous pages may seem repetitious at times, but they are intended to show Japanese mathematics educators' expectations of high school mathematics as well as the students' level of performance. As was stated in the introduction, one of the goals of the UECE Committee is to encourage more private universities to make use of the UECE in their selection of students. Because of the large number of applicants to all universities and because the function of the UECE is to test the basic material covered in the standard high school curriculum, many public and those private universities participating in the UECE will continue, at least in the near future, to use the UECE as a screening device to be followed by their own individual university examinations. The cram schools, Juku and Yobiko,* will also continue to serve the needs of the applicants.

For the purposes of comparison, sample 1991 university examinations from Tokyo University, Hokkaido University, Shiga University, and Shiga Medical University are included in the Appendix.

References

B. Duke, *The Japanese School: Lessons for Industrial America,* Praeger, New York, 1986.

H. Fujita, "The Present State and Current Problems of Mathematics Education at the Senior Secondary Level in Japan." *Proceedings of the International Conference on Mathematics Education,* University of Chicago Mathematics Project, 1985.

Robert Leestma, Robert L. August, Betty George, and Lois Peak, *Japanese Education Today,* U.S. Government Printing Office, Washington, D.C., 1987.

Robert Leestma and Herbert J. Walberg (eds.), *Japanese Educational Productivity,* Center for Japanese Studies, The University of Michigan, Ann Arbor, 1992.

Monbusho (Ministry of Education), *Jido, seito no Gakkogai Gakushu Katsudo ni Kansuru Jittai Chosa Hokokusho* (*A Survey Report on Students' Out-of-School Learning Activities*), 1985.

Monbusho (Ministry of Education), *Outline of Education in Japan,* 1989.

Monbusho (Ministry of Education), *Mombu Tokei Yoran* (*Statistical Abstract of Education, Science and Culture*), 1990.

Obunsha, "Zenkoku Daigaku Nyuushi Mondai Seikai Suugaku (Kokkouritsudaien)," 1991.

Obunsha, "Zenkoku Daigaku Naiyo Annai Go," *Keisetsu,* special issue, August, 1991.

T. P. Rohlen, *Japan's High Schools,* University of California Press, Berkeley, 1983.

Harold W. Stevenson and James W. Stigler, *The Learning Gap,* Summit Books, Simon Schuster, New York, 1992.

Momoru Tsukuda, *Yobiko Life: A Study of the Legitimation Process of Social Stratification in Japan,* Japan Research Monograph #11, Institute of East Asian Studies, University of California, Berkeley, 1991.

The National Center for University Examinations, "Report on the 1990 University Entrance Center Examination—Performance Results, Analysis, and Criticism of Exam Problems."

Merry White, *The Japanese Educational Challenge,* Free Press, New York, 1987.

Asahi Journal, Feb. 8, 1991.

* An excellent paper on Yobiko (preparatory schools) by Robert August is in *Japanese Educational Productivity.*

APPENDIX I

INDIVIDUAL UNIVERSITY EXAMINATIONS

The number of exam takers and the number of those who passed are provided for each examination. Among these four national universities, Tokyo University is recognized as having the highest ranking followed by Hokkaido University and then Shiga University and Shiga Medical University. Because of the exam calendar, a student can take at most two national university examinations. It is indeed crucial for applicants to make a careful personal assessment in order to target their application to an appropriate university. Often, the ratio of the number of applicants to the number accepted for a very difficult university may actually be smaller than that for a less difficult university which is a popular target.

The subjects referred to on the top of each examination are those covered in Japanese high school mathematics. The following is a brief description of each subject. For detailed topics in each subject, please see Fujita's paper.

Math I College algebra and a portion of trigonometry involving sine, cosine, and tangent, the law of sines, the law of cosines.

Math II Permutations, combinations, probability, statistics, vectors, differentiation and integration, sequences, exponential, logarithmic, and trigonometric functions, functions of electronic computers, algorithms, and flow charts.

Algebra/Geometry Conic sections, vectors in two and three dimensions, matrices.

Basic Analysis Sequences, mathematical induction, exponential, logarithmic, and trigonometric functions, derivatives and applications, integrals and applications.

Differentiation and Integration

Probability and Statistics

TOKYO UNIVERSITY

Examination A
Science

Date: February 25, 1991
Time: 150 minutes
Subjects: Math I, Algebra/Geometry, Basic Analysis, Calculus, Probability/Statistics
Number of exam takers: 2714
Number of those accepted: 1183
For the second stage exam, 139 out of 584 were accepted.

1. Given a regular pyramid with a triangular base set in a two-dimensional plane, select any edge of the three sides of the face touching the plane and use the edge as an axis to turn over the pyramid. Find the probability that, after repeating this action n times, the original face is again touching the plane.

2. Let a, b, and c be positive real numbers. In the xyz-space, consider the plane R consisting of points (x, y, z) satisfying the conditions

$$|x| \le a, \quad |y| \le b, \quad z = c.$$

Let P be the source of light on the plane

$$z = c + 1$$

moving along the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = c + 1$$

once around. Sketch and calculate the area of the shadow projected by the plane R on the xy-plane.

3. Let p be a constant. Consider the third-degree equation

$$x^3 - 3x - p = 0.$$

Let $f(p)$ be the product of the largest real root and the smallest real root. In case there is only one real root, $f(p)$ is defined as the square of that root.

(1) Find the minimum value of $f(p)$ for all p.

(2) Sketch the graph of $f(p)$ as a function of p.

4. (1) For natural numbers $n = 1, 2, \ldots$, prove that there exist polynomials $p_n(x)$, $q_n(x)$, such that

$$\sin n\theta = p_n(\tan\theta)\cos^n\theta,$$

$$\cos n\theta = q_n(\tan\theta)\cos^n\theta.$$

(2) Then, for $n > 1$, prove that one can establish the following identities:

$$p_n'(x) = nq_{n-1}(x),$$

$$q_n'(x) = -np_{n-1}(x).$$

5. On the xy-plane, a point (m, n), with m, n integers, is called a lattice point. Using each lattice point as a center, draw a circle of radius r. Any line with slope 2/5 intersects some of these circles. Find the minimum value of real numbers r having this property.

6. Let $f(x)$ be a continuous function defined for $x > 0$ such that $f(x_1) > f(x_2) > 0$ whenever $0 < x_1 < x_2$. Let

$$S(x) = \int_x^{2x} f(t)\,dt$$

and $S(1) = 1$. For any $a > 0$, the area bounded by the following is $3S(a)$:

(a) the line joining the origin and the point $(a, f(a))$;
(b) the line joining the origin and the point $(2a, f(2a))$;
(c) the curve $y = f(x)$.

(1) Express $S(x)$, $f(x) - 2f(2x)$ as a function of x.

(2) For $x > 0$, let

$$a(x) = \lim_{n\to\infty} 2^n f(2^n x).$$

Find the value of the integral

$$\int_x^{2x} a(t)\,dt.$$

(3) Determine the function $f(x)$.

Examination B
Humanities

Date: February 25, 1991
Time: 100 minutes
Subject: Math I, Algebra/Geometry, Basic Analysis
Number of exam takers: 1906
Number of those accepted: 642
For the second stage exam, 74 out of 331 were accepted.

1. Find the maximum and minimum values of

$$f(x) = x^3 - 2x^2 - 3x + 4$$

over the interval

$$-\frac{7}{4} \le x \le 3.$$

2. Consider the point P $(2, 0, 1)$ in xyz-space and the curve

$$z = y^2$$

in the yz-plane. As point Q moves along this curve, let R be the point of intersection of the line PQ (extended) and the xy-plane. Letting F be the graph of the set of points R, draw F in the xy-plane.

3. A rectangle $ABCD$ has side lengths 1 and a. Point E is the point of intersection of the two diagonals. Draw five circles centered at A, B, C, D, and E, each with radius r. Maximize r in such a way that the intersection of any two circles is empty. Let $S(a)$ be the total area of the five circles cut by the rectangle. Sketch the graph of

$$\frac{s(a)}{a}$$

as a function of a.

4. Given a regular pyramid V with a square base, there is a ball with its center on the bottom of the pyramid and tangent to all edges. If each edge of the pyramid base is a, find the following quantities:

(1) the height of V;

(2) the volume of the portion common to the ball and the pyramid.

Note: A regular pyramid has a square base adjoined to four isosceles triangles, with each edge of the square making up the base of one of the triangles.

HOKKAIDO UNIVERSITY

Examination A
Science I, Pre-medical and Pre-dental Applicants

Date: February 25, 1991
Time: 120 minutes
Subjects: Math I, Algebra/Geometry, Basic Analysis, Differential and Integral Calculus, Probability/Statistics
Number of exam takers: 1641
Number of those accepted: 636
For the second stage exam, 133 out of 838 were accepted.

1. Let α, β be two solutions of the second-degree equation in x:

$$x^2 - 2px + p^2 - 2p - 1 = 0.$$

Find a real value p such that

$$\frac{1}{2}\,\frac{(\alpha-\beta)^2 - 2}{(\alpha+\beta)^2 + 2}$$

is an integer.

2. For a real number t, let

$$x(t) = \frac{2^t + 2^{-t}}{2},$$
$$y(t) = \frac{2^t - 2^{-t}}{2}.$$

Let $M(t)$ be the centroid of the triangle whose vertices are $[x(t), y(t)]$, $[x(t+1), y(t+1)]$, and $[x(t+2), y(t+2)]$. Find the locus of $M(t)$ as t varies through all real numbers.

3. Given the curve $C\colon y = f(x)$, suppose that the tangent line at $P\,(x, y)$ is perpendicular to the line segment joining P and $Q\,(1, 0)$.

(1) Find a differential equation satisfied by $y = f(x)$.

(2) If $y = (-2/3)x + 5$ is tangent to the curve C, find the equation for the curve C.

4. Answer the following:

(1) Show that

$$\lim_{n\to\infty} \frac{1}{n}\left\{\sum_{k=n+1}^{2n} \log k - n\log n\right\} = \int_1^2 \log x\,dx.$$

(2) Find

$$\lim_{n\to\infty}\left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

5. Given three points O $(0,0,0)$, A $(\sqrt{2},0,0)$, and B $(0,\sqrt{2},0)$ and a sphere of radius $\sqrt{6}$ in xyz-space. The center of the sphere has a positive z-coordinate and the sphere is tangent to OA, OB, and AB.

(1) Find the coordinates of the center of the sphere.

(2) Find the volume of the portion of the sphere which is above the xy-plane.

Examination B
Science II, Science III and Fishery Applicants

Date: February 25, 1991
Time: 120 minutes
Subjects: Math I, Algebra/Geometry, Basic Analysis, Differential and Integral Calculus, Probability/Statistics
Number of exam takers: 2030
Number of those accepted: 764

1. Let α, β be two solutions of the second-degree equation in x:

$$x^2 - 2px + p^2 - 2p - 1 = 0.$$

Find a real value p such that

$$\frac{1}{2}\,\frac{(\alpha-\beta)^2 - 2}{(\alpha+\beta)^2 + 2}$$

is an integer.

2. Given three points on the plane O $(0,0)$, A $(1,0)$, and B $(-1,0)$, point P is moving on the plane satisfying the condition

$$\left(\overrightarrow{PA}, \overrightarrow{PB}\right) + 3\left(\overrightarrow{OA}, \overrightarrow{OB}\right) = 0.$$

(1) Find the locus of P.

(2) Find the maximum and minimum values of

$$|\overrightarrow{PA}|\cdot|\overrightarrow{PB}|.$$

3. Let $\triangle ABC$ be a triangle with $\angle A = 120^\circ$, $AB\cdot AC = 1$. Let D be the intersecting point of BC and the bisector of $\angle A$.

(1) If $AB = x$, express AD in terms of x.

(2) If x is a variable, find x such that AD is the maximum. Also find this maximum value.

4. Answer the following:

(1) Show that

$$\lim_{n\to\infty} \frac{1}{n}\left\{\sum_{k=n+1}^{2n} \log k - n\log n\right\} = \int_1^2 \log x\,dx.$$

(2) Find

$$\lim_{n\to\infty}\left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

5. Given three points O $(0,0,0)$, A $(3,0,0)$, B $(0,4,0)$, and a sphere of radius 2 in xyz-space. The center of the sphere has a positive z-coordinate and the three line segments OA, OB, and AB are tangent to the sphere. Find the coordinates of the center of the sphere.

Examination C
Humanities Applicants

Date: February 25, 1991
Time: 90 minutes
Subjects: Math I, Algebra/Geometry, Basic Analysis
Number of exam takers: 1953
Number of those accepted: 508
For the second stage exam, 69 out of 675 were accepted.

1. Let α, β be two solutions of the second-degree equation in x:

$$x^2 - 2px + p^2 - 2p - 1 = 0.$$

Find a real value p such that

$$\frac{1}{2}\,\frac{(\alpha-\beta)^2 - 2}{(\alpha+\beta)^2 + 2}$$

is an integer.

2. Given three points on the plane O $(0, 0)$, A $(1, 0)$, and B $(-1, 0)$, point P is moving on the plane satisfying the condition

$$\left(\overrightarrow{PA}, \overrightarrow{PB}\right) + 3\left(\overrightarrow{OA}, \overrightarrow{OB}\right) = 0.$$

(1) Find the locus of P.
(2) Find the maximum and minimum values of

$$|\overrightarrow{PA}| \cdot |\overrightarrow{PB}|.$$

3. n is a natural number greater than 2. Use mathematical induction to prove the inequality

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2.$$

4. Suppose $f(x)$ is a linear function of x. Let

$$F(x) = x\int_1^{2x+3} f(t)\,dt.$$

Determine $f(x)$ if $F(1) = 2$, $F'(0) = -10$.

SHIGA UNIVERSITY

Examination A
Education Division: Elementary, Secondary, Information Sciences

Date: February 25, 1991
Time: 90 minutes
Subjects: Math I, Algebra, Geometry, Basic Analysis
Number of exam takers: 1438
Number of those accepted: 388

1. Let α, β be two solutions of the second-degree equation

$$x^2 - px + 1 = 0.$$

Let α', β' be two solutions of the second-degree equation

$$x^2 - x + q = 0.$$

Express

$$(\alpha' - \alpha)(\alpha' - \beta)(\beta' - \alpha)(\beta' - \beta)$$

in terms of p, q.

2. Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ 1 & -2 \end{pmatrix}.$$

If $(A + B)^2 = A^2 + 2AB + B^2$, determine the values of a and b.

3. Graph the region bounded by the simultaneous inequalities

$$x - 2y^2 \geq 0$$

$$1 - x - |y| \geq 0.$$

Find the area of this region.

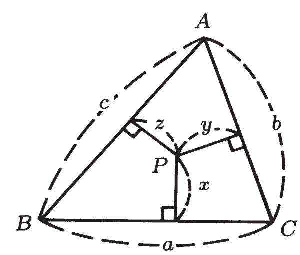
4. Sequence $\{a_n\}$ satisfies

$$a_1 + 2a_2 + 3a_3 + \cdots + na_n = \frac{n+1}{n+2} \quad (n \geq 1).$$

Find the sum

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

5. Let a, b, c be the sides of $\triangle ABC$ and S be the area. From an interior point P, draw the line perpendicular to each side with lengths x, y, and z, respectively, as is shown below:



(1) Express S in terms of a, b, c, x, y, and z.
(2) Let $P'(x, y, z)$ be a point corresponding to point P. Prove that as P moves within $\triangle ABC$, the graph of $P'(x, y, z)$ describes a triangle.
(3) Prove that if P is the center of gravity of $\triangle ABC$, then P' is also the center of gravity of the triangle obtained in (2).

Examination B
Economics Division

Date: March 5, 1991
Time: 100 minutes
Subject: Math I, Algebra/Geometry, Basic Analysis
Number of exam takers: 2803
Number of those accepted: 686

1. Consider the graphs of the function

$$y = \sqrt{2x - 1}$$

and the straight line $y = x + k$.

Discuss the number of points of intersection versus the change in the value of k.

2. In $\triangle ABC$, Q is a point on AC such that $AQ : QC = 3 : 4$. P is a point on BQ such that $BP : PQ = 7 : 2$. Let O be any point. Answer the following questions.

(1) Express
$$\overrightarrow{OQ} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OC},$$
where α and β are rational numbers.

(2) Express
$$\overrightarrow{OP} = l\overrightarrow{OA} + m\overrightarrow{OB} + n\overrightarrow{OC}.$$
where l, m, and n are rational numbers.

3. Let $\angle C$ be the 90° angle in the right triangle ABC. Set the hypotenuse AB equal to a.

(1) If D is the point of tangency of the inscribed circle and the hypotenuse AB, prove that
$$AD = \frac{1}{2}(AB + AC - BC).$$

(2) If θ is $\frac{1}{2}\angle A$, express the radius r of the inscribed circle in terms of a and θ.

(3) Find the maximum value of r.

4. For each x, let $f(x)$ be equal to the smaller of the two values $(x-a)^2$ and $(x-a-2)^2$. Let
$$F(a) = \int_0^1 f(x)\,dx.$$
Find the maximum and the minimum of $F(a)$ with $-2 \le a \le 2$.

SHIGA MEDICAL UNIVERSITY

Medical School Examination

Date: March 5, 1991
Time: 120 minutes
Subject: Math I, Algebra/Geometry, Basic Analysis, Calculus
Number of exam takers: 531
Number of those accepted: 140

1. (1) Given $a^2 \ge b$, where a, b are natural numbers, prove that the necessary and sufficient condition for
$$\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$$
to be a natural number is that there exists a natural number n such that
$$n^2 < a \le 2n^2 \quad \text{and} \quad b = 4n^2(a-n^2)$$

(2) Find all values of natural numbers b such that
$$\sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}}$$
is a natural number.

2. $\triangle ABC$ varies with time t. Suppose in $\triangle ABC$, $\angle A$ is an acute angle θ and the area is fixed. Let the lengths of the three sides be $a = BC$, $b = CA$, and $c = AB$.

(1) Express the rate of change of θ with respect to t
$$\frac{d\theta}{dt}$$
in terms of
$$b,\quad c,\quad \frac{db}{dt},\quad \frac{dc}{dt},\ \text{and}\ \theta.$$

(2) Express the rate of change of a with respect to t
$$\frac{da}{dt}$$
in terms of
$$b,\quad c,\quad \frac{db}{dt},\quad \frac{dc}{dt},\ \text{and}\ \theta.$$

(3) Suppose $\triangle ABC$ is an equilateral triangle with each side equal to 10 and
$$\frac{db}{dt} = 2, \qquad \frac{dc}{dt} = -1.$$
Find the rate of change for θ and for a with respect to t.

3. Let AB be a line segment of 1 unit length moving in such a way that A $(s, 0)$ with $0 \le s \le 1$ is a point on the x-axis and B $(0, t)$ with $0 \le t \le 1$ is a point on the y-axis.

(1) Express in terms of s and x_0 the y-coordinate of the point of intersection of AB and the line $x = x_0$ where $0 \le x_0 \le s$.

(2) Use the inequalities to express the interval of possible values of x and the interval of possible values of y where (x, y) is a point on AB.

4. Let l be the line of intersection of two planes:
$$\Pi_1\colon ax + y + z = a$$
$$\Pi_2\colon x - ay + az = -1.$$

(1) Find a direction vector of the line l.

(2) The line l describes a surface as the real number a varies. Let (x, y) be the point of intersection of the surface and the plane $z = t$. Find an equation which gives the relation between x and y.

(3) Find the volume bounded by the two planes, $x = 0$ and $x = 1$, and the surface obtained in (2) as t varies.

APPENDIX II

SOLUTIONS TO TOKYO UNIVERSITY EXAMS

Examination A Science

1. For $n = 0, 1, 2, \ldots$, define
$$P_n = \text{the probability of having the first face touching the plane after } n \text{ operations.}$$
The sequence $\{P_n\}$ satisfies the recursive relation:
$$P_{n+1} = (1 - P_n) \times \frac{1}{3}$$

This can be written as

$$P_{n+1} - \frac{1}{4} = -\frac{1}{3}\left(P_n - \frac{1}{4}\right).$$

That is, $\{P_n - \frac{1}{4}\}$ is a geometric progression with the ratio $-\frac{1}{3}$:

$$P_n - \frac{1}{4} = \left(-\frac{1}{3}\right)^n \left(P_0 - \frac{1}{4}\right).$$

From the definition of P_n, $P_0 = 1$. Therefore,

$$P_n = \frac{1}{4} + \frac{3}{4}\left(-\frac{1}{3}\right)^n.$$

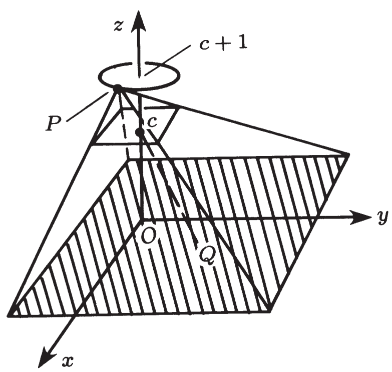
2. We shall rescale the x-axis by $1/a$ times, and the y-axis by $1/b$ times so that we first consider the case with the plane

$$R\colon |x| \leq 1,\ |y| \leq 1,\ z = c$$

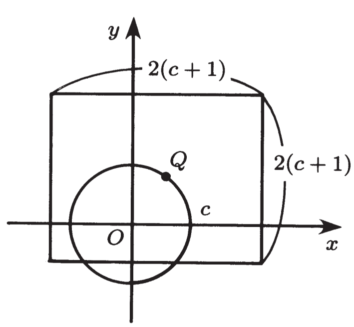
and the source light is moving along the circle

$$x^2 + y^2 = 1, \quad z = c + 1.$$

Then we adjust accordingly afterwards.



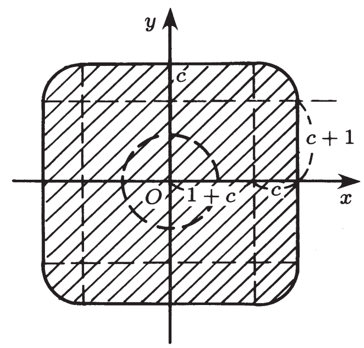
First let us fix the point $P\,(\alpha, \beta, c+1)$. The image of R on the xy-plane is a rectangle enlarging R by $(c+1)$ times with the center $Q\,(-c\alpha, -c\beta, 0)$ which is the corresponding image of P on the xy-plane.



Next as P moves along the circle, Q moves along the circle $x^2 + y^2 = c^2$ on the xy-plane.

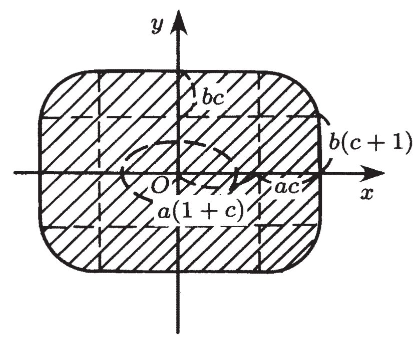
The area covered by the squares of $2(c+1)$ by $2(c+1)$ with a center on $x^2 + y^2 = c^2$ is

$$\pi c^2 + 8c(c+1) + 4(c+1)^2.$$



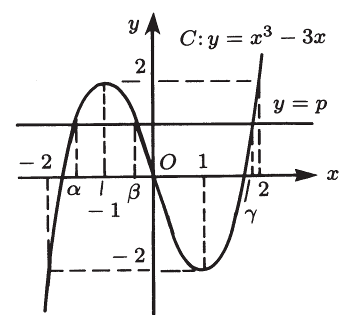
Adjusting to the case of the ellipse, the required area is

$$ab[(\pi + 12)c^2 + 16c + 4].$$



3. (1) From the graph of $y = x^3 - 3x$, the number of real roots is

$$\begin{cases} 3 & \text{when } |p| \leq 2, \\ 1 & \text{when } |p| > 2. \end{cases}$$



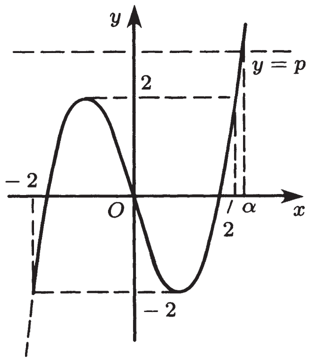
Case (i) $|p| \leq 2$.
Let α, β, γ be three real roots with $\alpha \leq \beta \leq \gamma$. Then

$$f(p) = \alpha\gamma.$$

From the relationship between the roots and coefficients of a polynomial equation, $\alpha\beta\gamma = p$, and we have

$$f(p) = \frac{p}{\beta} = \frac{\beta^3 - 3\beta}{\beta} = \beta^2 - 3$$

with $-1 \leq \beta \leq 1$.



The minimum value of $f(p)$ for $|p| \leq 2$ is when $\beta = 0$. That is, the minimum value is -3 when $p = 0$.

Case (ii) $|p| > 2$.

Let α be the single real root. Then

$$f(p) = \alpha^2.$$

Since $\alpha < -2$ or $\alpha > 2$, $f(p) > 4$. Therefore, the minimum value of $f(p)$ is -3.

(2) Use the same symbols as in (1).

Case (i) $|p| < 2$.

As p increases, β decreases from 1 to -1 and $f(p) = \beta^2 - 3$ varies as indicated in the chart below.

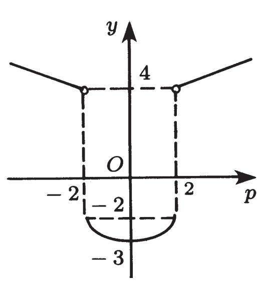
p	-2		0		2
β	1	↘	0	↘	-1
$f(p)$	-2	↘	-3	↗	-2

Case (ii) $|p| > 2$.

As p increases, $|\alpha|$ increases and $f(p) = \alpha^2$ varies as indicated in the chart below.

p	$-\infty$	-2 2	
α	$-\infty$	↗ -2 2 ↗	∞
$f(p)$	$-\infty$	↘ 4 4 ↗	∞

The following is a rough sketch of the graph of $f(p)$.



4. (1) Proof by mathematical induction.

First note that

$$\begin{cases} \sin\theta = \tan\theta \cdot \cos\theta \\ \cos\theta = 1 \cdot \cos\theta \end{cases}$$

so for $n = 1$, take $p_1(x) = x$, and $q_1 = 1$.

For $n \geq 2$ suppose there are polynomials $p_{n-1}(x)$, q_{n-1} such that

$$\sin(n-1)\theta = p_{n-1}(\tan\theta)\cos^{n-1}\theta$$

$$\cos(n-1)\theta = q_{n-1}(\tan\theta)\cos^{n-1}\theta$$

From the addition formula, we have

$$\sin n\theta = \sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta$$

$$= p_{n-1}(\tan\theta)\cos^n\theta + q_{n-1}(\tan\theta)\cos^n\theta\tan\theta$$

$$\cos n\theta = \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta$$

$$= q_{n-1}(\tan\theta)\cos^n\theta - p_{n-1}(\tan\theta)\cos^n\theta\tan\theta$$

therefore, polynomials $p_n(x)$ and $q_n(x)$ are

$$\left.\begin{aligned} p_n(x) &= p_{n-1}(x) + xq_{n-1}(x) \\ q_n(x) &= q_{n-1}(x) - xp_{n-1}(x) \end{aligned}\right\} \qquad (1)$$

and

$$\sin n\theta = p_n(\tan\theta)\cos^n\theta$$

$$\cos n\theta = q_n(\tan\theta)\cos^n\theta$$

(2) Proof by induction also.

First note that

$$\begin{cases} p_1(x) = x, \quad q_1(x) = 1 \\ p_2(x) = 2x, \quad q_2(x) = 1 - x^2 \end{cases}$$

which establishes the required identity for $n = 2$.

For $n > 2$, suppose (2) holds for $m - 1$, i.e.,

$$p'_{m-1}(x) = (m-1)q_{m-2}(x)$$

$$q'_{m-1}(x) = -(m-1)p_{m-2}(x)$$

From equation (1) above

$$p'_m(x) = p'_{m-1}(x) + xq'_{m-1}(x) + q_{m-1}(x)$$

$$= (m-1)\{q_{m-2}(x) - xp_{m-2}(x)\} + q_{m-1}(x)$$

$$= (m-1)q_{m-1}(x) + q_{m-1}(x)$$

$$= mq_{m-1}(x)$$

$$q'_m(x) = q'_{m-1}(x) - xp'_{m-1}(x) - p_{m-1}(x)$$

$$= -(m-1)p_{m-2}(x) - (m-1)xq_{m-2}(x) - p_{m-1}(x)$$

$$= -(m-1)\{p_{m-2}(x) + xq_{m-2}(x)\} - p_{m-1}(x)$$

$$= -(m-1) \cdot p_{m-1}(x) - p_{m-1}(x)$$

$$= -mp_{m-1}(x).$$

This establishes (2) for $n = m$.

5. A line with the slope $\frac{2}{5}$ can be expressed as

$$2x - 5y - k = 0 \quad (k \text{ a constant})$$

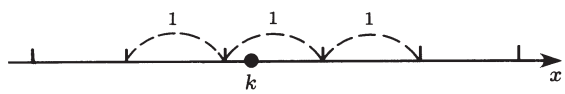
This line and the circle with radius r and center at (m, n) will have a common point if

$$\frac{|2m - 5n - k|}{\sqrt{2^2 + (-5)^2}} \leq r,$$

i.e.,

$$|2m - 5n - k| \leq \sqrt{29}r. \qquad (1)$$

Therefore, we should look for a condition on r to guarantee that given any real number k, there is a pair of whole numbers m, n such that (1) is true. As the pair m, n varies, $2m - 5n$ varies through all integers. In fact for any integer N, $N = 2 \cdot (3N) - 5 \cdot N$. Thus the above condition is equivalent to "for any real number k, there is an integer N such that $|N - k| \leq \sqrt{29}r$." That is, "On the real line, for any real number k, there is an integral point N such that the distance between the point k and the point N is less than or equal to $\sqrt{29}r$."



Since the integral points on the real line are 1 unit apart, it follows that $\sqrt{29}r \geq \frac{1}{2}$, i.e.,

$$r \geq \frac{1}{2\sqrt{29}} = \frac{\sqrt{29}}{58}.$$

Therefore, the minimum value of r is $\sqrt{29}/58$.

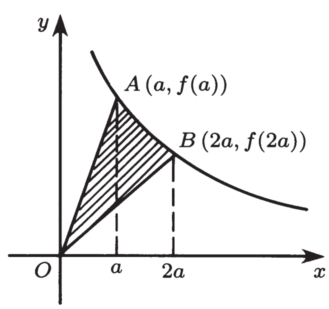
6. (1)

Area bounded by OA, OB, and $f(x)$

$$= 3S(a)$$

$$= \frac{1}{2}a \cdot f(a) + S(a) - \frac{1}{2} \cdot 2a \cdot f(a).$$

$\therefore S(a) = \frac{1}{4}(a \cdot f(a) - 2a \cdot f(2a))$ for any $a > 0$.



Replacing a by x:

$$S(x) = \frac{1}{4}\left(x\,f(x) - 2x\,f(2x)\right). \qquad (1)$$

On the other hand, differentiating

$$S(x) = \int_x^{2x} f(t)\,dt,$$

we have

$$S'(x) = 2f(2x) - f(x). \qquad (2)$$

From (1) and (2), we have a differential equation

$$xS'(x) = -4S(x) \quad x > 0.$$

Solving with the initial condition $S(1) = 1$, we have

$$S(x) = \frac{1}{x^4}. \qquad (3) \text{ answer}$$

Also (2) and (3) together give

$$f(x) - 2f(2x) = \frac{4}{x^5}. \qquad (4) \text{ answer}$$

(2) From (4) above, given any $t > 0$ and any integer k

$$f(2^k t) - 2f(2^{k+1}t) = \frac{4}{(2^k t)^5}.$$

$$\therefore 2^k f(2^k t) - 2^{k+1} f(2^{k+1} t) = \frac{4}{2^{4k} t^5}.$$

Summing the above equations for $k = 0, 1, 2, \ldots, n-1$, we have

$$\sum_{k=0}^{n-1}\{2^k f(2^k t) - 2^{k+1} f(2^{k+1} t)\} = \frac{4}{t^5}\sum_{k=0}^{n-1}\frac{1}{2^{4k}}$$

and hence

$$f(t) - 2^n f(2^n t) = \frac{4}{t^5}\,\frac{1 - \frac{1}{16^n}}{1 - \frac{1}{16}}.$$

As $n \to \infty$, we see that

$$f(t) - a(t) \to \frac{64}{15}\frac{1}{t^5}. \qquad (5)$$

For any $x > 0$, take the definite integral of both sides of (5) from $t = x$ to $t = 2x$,

$$S(x) - \int_x^{2x} a(t)\,dt = \frac{64}{15}\int_x^{2x}\frac{dt}{t^5}.$$

$$\therefore \int_x^{2x} a(t)\,dt = \frac{1}{x^4} - \frac{64}{15}\left[-\frac{1}{4t^4}\right]_x^{2x} = 0. \qquad \text{answer}$$

(3) From the definition of $a(x)$, $a(x) \geq 0$ for all $x > 0$. Also because $a(x)$ is continuous following (5),

$$a(x) = 0$$

for all x in order to establish the result of (2). This implies

$$f(x) = \frac{64}{15} \cdot \frac{1}{x^5}, \qquad x > 0. \qquad \text{answer}$$

Examination B
Humanities

1. $f'(x) = 3x^2 - 4x - 3$. Let $\alpha = (2 - \sqrt{13})/3$ and $\beta = (2 + \sqrt{13})/3$, then $-\frac{7}{4} < \alpha < \beta < 3$.

x	$-\frac{7}{4}$		α		β		3
$f'(x)$		$+$		$-$		$+$	
$f(x)$		↗		↘		↗	

Using the relation

$$f(x) = f'(x)\left(\frac{1}{3}x - \frac{2}{9}\right) - \frac{2}{9}(13x - 15)$$

we calculate

$$f(\alpha) = -\frac{2}{9}(13\alpha - 15) = \frac{38 + 26\sqrt{13}}{27}$$

$$f(\beta) = -\frac{2}{9}(13\beta - 15) = \frac{38 - 26\sqrt{13}}{27}.$$

Also,

$$f\left(-\frac{7}{4}\right) = -\frac{143}{64}, \qquad f(3) = 4$$

$$\frac{38+26\sqrt{13}}{27} > 7, \qquad \frac{38-26\sqrt{13}}{27} > -\frac{143}{64}.$$

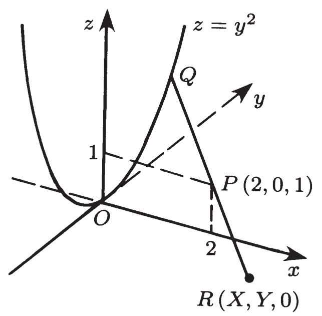
$\therefore$ The maximum value of f is

$$\max(f(\alpha), f(3)) = \frac{38+26\sqrt{13}}{27}. \qquad \text{answer}$$

The minimum value of f is

$$\min\left(f\left(-\frac{7}{4}\right), f(\beta)\right) = -\frac{143}{64}. \qquad \text{answer}$$

2. Let Q be the intersection of the line joining P and the point $R\,(X, Y, 0)$ on the xy-plane.



Solving the simultaneous equations

$$\begin{cases} \dfrac{x-2}{X-2} = \dfrac{y}{Y} = \dfrac{z-1}{-1}, \\ x = 0 \end{cases}$$

we have

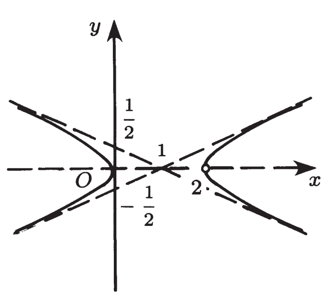
$$Q\left(0, \frac{-2Y}{X-2}, \frac{X}{X-2}\right).$$

If Q is on the parabola $z = y^2$,

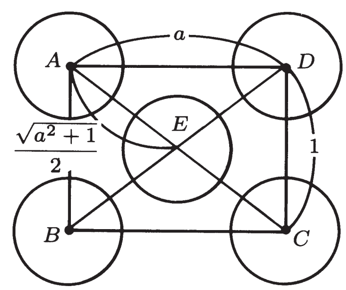
$$\frac{X}{X-2} = \left(\frac{-2Y}{X-2}\right)^2.$$

$$\therefore X(X-2) - 4Y^2 = 0 \quad \text{and} \quad X \neq 2.$$

All points $(X, Y, 0)$ satisfying the above are points of the graph of $(x-1)^2 - 4y^2 = 1$ on the xy-plane with one point $(2, 0)$ removed.

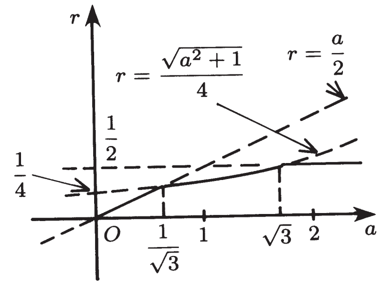


3. Let $AB = 1$, $AD = a$ in the graph below.



In order that any two circles have empty intersection, $2r \leq 1$ and $2r \leq a$. Also $2r \leq \sqrt{a^2+1}/2$. Therefore, the maximum value of r satisfying all of these conditions is

$$r = \min\left(\frac{1}{2}, \frac{a}{2}, \frac{\sqrt{1+a^2}}{4}\right)$$



(i) when $0 < a \leq \dfrac{1}{\sqrt{3}}$, $r = \dfrac{a}{2}$

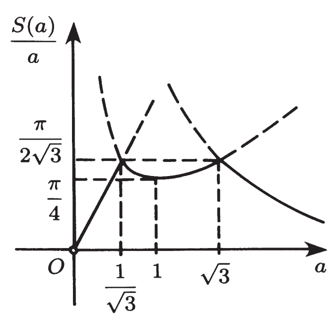
(ii) when $\dfrac{1}{\sqrt{3}} < a \leq \sqrt{3}$, $r = \dfrac{\sqrt{a^2+1}}{4}$

(iii) when $\sqrt{3} < a$, $r = \dfrac{1}{2}$

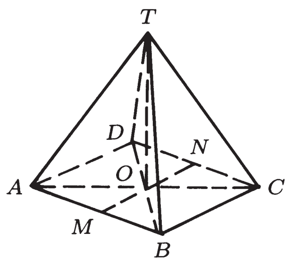
Substituting the above result in $S(a) = 2\pi r^2$, we have

$$\frac{S(a)}{a} = \begin{cases} \dfrac{\pi}{2}a, & \text{if } 0 < a \leq \dfrac{1}{\sqrt{3}} \\ \dfrac{\pi}{8} \cdot \dfrac{a^2+1}{a}, & \text{if } \dfrac{1}{\sqrt{3}} < a < \sqrt{3} \\ \dfrac{\pi}{2} \cdot \dfrac{1}{a}, & \text{if } \sqrt{3} \leq a \end{cases}$$

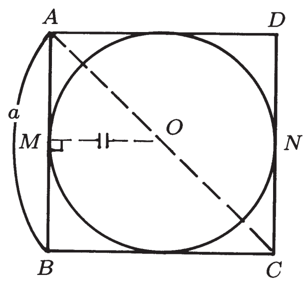
and the following graph.



4. As shown below, let T be the vertex of V and M, N be the midpoints of AB, CD, respectively.



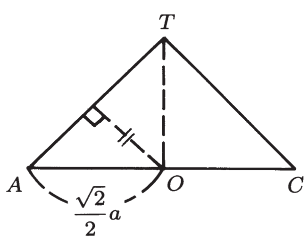
The center of the ball is on the bottom of V and the ball is tangent to four sides of the square. Therefore, the center of the ball is the center of the square and the radius is $a/2$.



Next, consider the cross section TAC. From the fact that the ball centered at O is tangent to TA and the fact that $AO = (\sqrt{2}/2)a$, we have $\angle TAO = 45°$.

$$\therefore TO = AO = \frac{\sqrt{2}}{2}a.$$

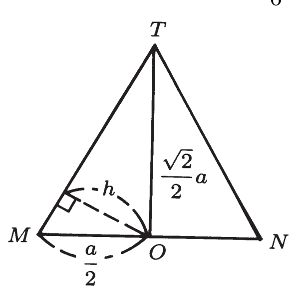
(1) It follows that the height of V is $\frac{\sqrt{2}}{2}a$.



(2) To find the volume K of that portion of the ball cut by one lateral side of V; the distance h from the center O to a lateral side of V as shown below is

$$h \cdot \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{\sqrt{2}a}{2}\right)^2} = \frac{a}{2} \cdot \frac{\sqrt{2}}{2}a$$

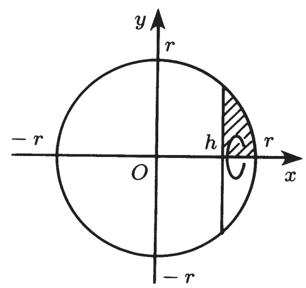
$$\therefore h = \frac{\sqrt{6}}{6}a.$$



Therefore,

$$K = \pi \int_h^{a/2} \left(\frac{a^2}{2^2} - x^2\right) dx$$

$$= \frac{1}{8}\left(\frac{2}{3} - \frac{7\sqrt{6}}{27}\right)\pi a^3.$$



The required volume is

$$\frac{2\pi}{3}\left(\frac{a}{2}\right)^3 - \frac{1}{8}\left(\frac{2}{3} - \frac{7\sqrt{6}}{27}\right)\pi a^3 \times 4$$

$$= \left(\frac{7\sqrt{6}}{54} - \frac{1}{4}\right)\pi a^3. \qquad \text{answer}$$

SOLUTIONS TO HOKKAIDO UNIVERSITY EXAMS

Examination A
Science I, Pre-medical and Pre-dental Applicants

1. From the relationship between the roots and the coefficients, we have

$$\alpha + \beta = 2p$$

$$\alpha\beta = p^2 - 2p - 1$$

$$\therefore (\alpha - \beta)^2 - 2 = (\alpha + \beta)^2 - 4\alpha\beta - 2 = 8p + 2$$

$$(\alpha + \beta)^2 + 2 = 4p^2 + 2$$

$$\therefore \frac{1}{2}\frac{(\alpha - \beta)^2 - 2}{(\alpha + \beta)^2 + 2} = \frac{4p + 1}{4p^2 + 2} = n \quad \text{(an integer)}$$

$$4np^2 - 4p + 2n - 1 = 0.$$

Since p is a real number, either $n = 0$,
or $n \neq 0$ and the discriminant $= 4 - 4n(2n - 1) \geq 0$.

$$\therefore 2n^2 - n - 1 \leq 0$$

$$(2n + 1)(n - 1) \leq 0$$

$$-\frac{1}{2} \leq n \leq 1$$

$$\therefore n = 1$$

when $n = 0$, $p = -\frac{1}{4}$. answer

When $n = 1$,

$$4p^2 - 4p + 1 = 0$$
$$(2p-1)^2 = 0$$
$$p = \frac{1}{2}. \qquad \text{answer}$$

2. Let $G\ (X, Y)$ be the center of gravity.

$$\begin{aligned} X &= \frac{1}{3}\{x(t) + x(t+1) + x(t+2)\} \\ &= \frac{1}{6}(2^t + 2^{t+1} + 2^{t+2} + 2^{-t} + 2^{-(t+1)} + 2^{-(t+2)}) \\ &= \frac{1}{6}\left[2^{t+1}\left(\frac{1}{2} + 1 + 2\right) + 2^{-t-1}\left(2 + 1 + \frac{1}{2}\right)\right] \\ &= \frac{7}{12}\{2^{t+1} + 2^{-t-1}\} \geq \frac{7}{6} \end{aligned}$$

since $y + y^{-1} \geq 2$ when $y > 0$.

$$\begin{aligned} Y &= \frac{1}{3}\{y(t) + y(t+1) + y(t+2)\} \\ &= \frac{7}{6}\left(2^t - \frac{1}{4}2^{-t}\right) \\ &= \frac{7}{12}\left(2^{t+1} - 2^{-t-1}\right). \end{aligned}$$

(Since $2^t > 0$, Y assumes all real values.)

$$X + Y = \frac{7}{6} \cdot 2^{t+1}$$
$$X - Y = \frac{7}{6} \cdot 2^{-t-1}$$
$$\therefore X^2 - Y^2 = \frac{49}{36}.$$

The locus is the portion of

$$x^2 - y^2 = \frac{49}{36} \qquad \text{where } x \geq \frac{7}{6}. \qquad \text{answer}$$

3. (1) The slope of PQ is $y/(x-1)$. The slope of the tangent of C at P is dy/dx. The line PQ is perpendicular to the tangent, so

$$\frac{dy}{dx} \cdot \frac{y}{x-1} = -1, \quad y\,dy = -(x-1)\,dx \qquad \text{answer}$$

(2) $\int y\,dy = \int -(x-1)\,dx \;\therefore\; (x-1)^2 + y^2 = a\ (> 0)$. Hence the curve C is a circle with center at $(1, 0)$ and the radius $\sqrt{a}$, and

$$y = f(x) = \pm\sqrt{a - (x-1)^2}.$$

The condition that the curve C is tangent to the line

$$y = -\frac{2}{3}x + 5 \qquad (\text{i.e., } 2x + 3y - 15 = 0) \qquad (1)$$

is that the distance from the point $(1, 0)$ to line (1) is $\sqrt{a}$. Therefore,

$$\frac{13}{\sqrt{4+9}} = \sqrt{a} \qquad \text{and} \qquad \therefore a = 13$$

The fact that the y-intercept of (1) is positive implies that

$$y = \sqrt{13 - (x-1)^2}. \qquad \text{answer}$$

4. (1) Let

$$I = \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{k=n+1}^{2n} \log k - n\log n\right\},$$

then

$$\begin{aligned} I &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=n+1}^{2n} (\log k - \log n) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=n+1}^{2n} \log \frac{k}{n}. \end{aligned}$$

This is the definite integral of $\log x$ over $[1, 2]$ obtained by dividing the interval into n equal portions.

$$\therefore I = \int_1^2 \log x\,dx.$$

(2) Let

$$P_n = \left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

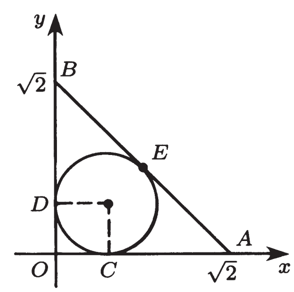
Then

$$\begin{aligned} \log P_n &= \frac{1}{n}\log\frac{(n+1)(n+2)\cdots(2n)}{n^n} \\ &= \frac{1}{2}\sum_{k=n+1}^{2n} \log\frac{k}{n}. \end{aligned}$$

$$\begin{aligned} \therefore \lim_{n\to\infty} \log P_n = I &= \int_1^2 \log x\,dx = (x\log x - x)_1^2 \\ &= 2\log 2 - 1 = \log\frac{4}{e}. \end{aligned}$$

$$\therefore \lim_{n\to\infty} P_n = \frac{4}{e}.$$

5. (1) $\triangle OAB$ is on the xy-plane.



The cross section of the sphere tangent to three sides is the inscribed circle of $\triangle OAB$. Let C, D, and E be points of tangency on OA, OB, and OC, respectively. Let r be the radius of the inscribed circle. Then

$$OA = OB = \sqrt{2}, \quad AB = 2$$

together with

$$OC = OD = r, \quad CA = CE, \quad BD = BE$$

imply

$$r = OC = OD = \frac{1}{2}(OA + OB - AB) = \sqrt{2} - 1.$$

The distance between the center of the sphere (r, r, z), $(z > 0)$, and $C\ (r, 0, 0)$ is the radius of the sphere, so

$$r^2 + z^2 = 6.$$

$$\therefore z^2 = 3 + 2\sqrt{2} = (\sqrt{2}+1)^2.$$

$$\therefore z = \sqrt{2} + 1.$$

The center of the sphere is

$$(\sqrt{2}-1, \sqrt{2}-1, \sqrt{2}+1). \qquad \text{answer}$$

(2) The portion of the sphere above the xy-plane is that portion of $x^2+y^2+z^2=6$ and $x \le \sqrt{2}+1$. The volume of this portion is

$$\begin{aligned}\int_{-\sqrt{6}}^{\sqrt{2}+1} \pi(6-x^2)\,dx &= \pi\left[6x - \frac{1}{3}x^3\right]_{-\sqrt{6}}^{\sqrt{2}+1} \\ &= \pi\left\{6\left(\sqrt{2}+1\right) - \frac{2\sqrt{2}+6+3\sqrt{2}+1}{3} + 6\sqrt{6} - 2\sqrt{6}\right\} \\ &= \frac{\pi}{3}\left(11 + 13\sqrt{2} + 12\sqrt{6}\right). \qquad \text{answer}\end{aligned}$$

Examination B Science II, Science III, and Fishery Applicants

1. See Exam A, #1.

2. (1) Let $P = (x, y)$. Then $\overrightarrow{PA} = (1-x, -y)$, $\overrightarrow{PB} = (-1-x, -y)$, therefore, the inner products are

$$(\overrightarrow{PA}, \overrightarrow{PB}) = (x^2-1) + y^2 = x^2 + y^2 - 1$$

$$(\overrightarrow{OA}, \overrightarrow{OB}) = -1.$$

So

$$(\overrightarrow{PA}, \overrightarrow{PB}) + 3(\overrightarrow{OA}, \overrightarrow{OB}) = 0$$

gives

$$x^2 + y^2 = 4.$$

Answer. The required locus is the circle with center at $(0,0)$ and radius 2.

(2)
$$\begin{aligned}|\overrightarrow{PA}|^2 \cdot |\overrightarrow{PB}|^2 &= \{(1-x)^2 + y^2\}\{(1+x)^2 + y^2\} \\ &= \{1 + x^2 + y^2 - 2x\}\{1 + x^2 + y^2 + 2x\} \\ &= \{5 - 2x\}\{5 + 2x\} \\ &= 25 - 4x^2.\end{aligned}$$

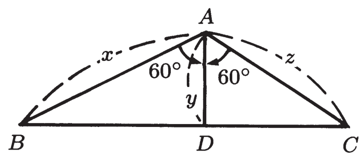
Since $0 \le x^2 \le 4$, $9 \le 25 - 4x^2 \le 25$.

$$\therefore 3 \le |\overrightarrow{PA}| \cdot |\overrightarrow{PB}| \le 5.$$

Answer. The maximum value is 5, and the minimum value is 3.

3. (1) Let $AD = y$, and $AC = z$, then

$$AB \cdot AC = xz = 1. \qquad (1)$$



$$\angle BAD = \angle CAD = 60^\circ$$

$$\triangle ABD + \triangle CAD = \triangle ABC$$

$$\frac{1}{2}xy\sin 60^\circ + \frac{1}{2}yz\sin 60^\circ = \frac{1}{2}xz\sin 120^\circ$$

$$\therefore y(x+z) = xz$$

Using (1)

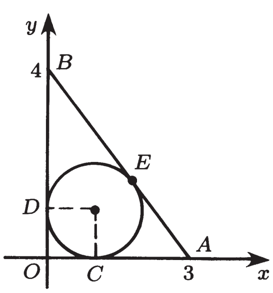
$$\begin{aligned}AD = y &= \frac{xz}{x+z} \\ &= \frac{1}{x+z} = \frac{x}{x^2+xz} = \frac{x}{x^2+1} \qquad \text{answer}\end{aligned}$$

(2)
$$y = \frac{1}{x+\frac{1}{x}} \le \frac{1}{2\sqrt{x\cdot\frac{1}{x}}} = \frac{1}{2}.$$

The equality holds when $x = 1/x$, i.e., $x = 1$. Therefore, the maximum value is $1/2$ when $x = 1$.

4. See Exam A, #4.

5. $\triangle OAB$ is on the xy-plane. The cross section of the sphere tangent to the three sides is the inscribed circle of $\triangle OAB$. Let C, D, and E be points of tangency on OA, OB, and OC, respectively. Let r be the radius of the inscribed circle.



Then $OA = 3$, $OB = 4$, and $AB = 5$, which, together with $OC = OD = r$, $CA = CE$, and $BD = BE$, imply

$$r = OC = OD = \frac{1}{2}(OA + OB - AB) = 1.$$

The distance between the center of the sphere $(1, 1, z)$ $(z > 0)$ and $C\ (1, 0, 0)$ is the radius of the sphere

$$1^2 + z^2 = 2^2 \qquad \therefore z = \sqrt{3}.$$

Answer. Therefore, the center of the sphere is $(1, 1, \sqrt{3})$.

Examination C
Humanities Applicants

1. See Exam A, #1.

2. See Exam B, #2.

3. To prove

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2, \quad n \geq 2 \qquad (1)$$

by mathematical induction:

(i) For $n = 2$, inequality (1) is $1/\sqrt{2} < 2\sqrt{2} - 2$.

$$(2\sqrt{2} - 2) - \frac{1}{\sqrt{2}} = \frac{3 - 2\sqrt{2}}{\sqrt{2}} > 0$$

establishes (1) for $n = 2$.

(ii) Suppose (1) holds for $n = k$. That is,

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} < 2\sqrt{k} - 2.$$

Then

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} - 2 + \frac{1}{\sqrt{k+1}}.$$

Now

$$\begin{aligned}
(2\sqrt{k+1} - 2) - \left(2\sqrt{k} - 2 + \frac{1}{\sqrt{k+1}}\right)
&= 2(\sqrt{k+1} - \sqrt{k}) - \frac{1}{\sqrt{k+1}} \\
&= \frac{2}{\sqrt{k+1} + \sqrt{k}} - \frac{1}{\sqrt{k+1}} \\
&= \frac{2\sqrt{k+1} - (\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1}(\sqrt{k+1} + \sqrt{k})} \\
&= \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k+1}(\sqrt{k+1} + \sqrt{k})} > 0.
\end{aligned}$$

This establishes (1) for $n = k+1$. By the principle of mathematical induction, (1) is established for all natural numbers $n \geq 2$.

4. Let

$$f(x) = ax + b.$$

Then

$$F(x) = x \int_1^{2x+3} (at + b)\, dt = x \int_1^{2x+3} f(t)\, dt$$

and

$$F'(x) = x \cdot 2f(2x+3) + \int_1^{2x+3} f(t)\, dt.$$

$$F(1) = \int_1^5 (at + b)\, dt = \left(\frac{at^2}{2} + bt\right)_1^5 = 12a + 4b.$$

$$F'(0) = \int_1^3 f(t)\, dt = 4a + 2b.$$

$$\left.\begin{aligned} F(1) &= 2 = 12a + 4b \\ F'(0) &= -10 = 4a + 2b \end{aligned}\right\} \Longrightarrow a = \frac{11}{2},\ b = -16$$

$$\therefore f(x) = \frac{11}{2}x - 16. \qquad \text{answer}$$

SOLUTIONS TO SHIGA UNIVERSITY EXAMS

Examination A
Education Division: Elementary, Secondary, Information Sciences

1. Since α' and β' are the roots of $x^2 - x + q = 0$,

$$x^2 - x + q = (x - \alpha')(x - \beta').$$

When we substitute $x = \alpha, \beta$,

$$\alpha^2 - \alpha + q = (\alpha - \alpha')(\alpha - \beta') = (\alpha' - \alpha)(\beta' - \alpha)$$

$$\beta^2 - \beta + q = (\beta - \alpha')(\beta - \beta') = (\alpha' - \beta)(\beta' - \beta).$$

Multiplying the two equations above, we have

$$\begin{aligned}
&(\alpha' - \alpha)(\beta' - \alpha)(\alpha' - \beta)(\beta' - \beta) \\
&\quad = (\alpha^2 - \alpha + q)(\beta^2 - \beta + q) \\
&\quad = q^2 + \left[(\alpha+\beta)^2 - 2\alpha\beta - (\alpha+\beta)\right] q \\
&\qquad + \alpha\beta\left[\alpha\beta - (\alpha+\beta) + 1\right] \\
&\quad = q^2 + (p^2 - 2 - p)q + 1(1 - p + 1) \\
&\quad = q^2 + p^2q - 2q - pq - p + 2 \qquad \text{answer}
\end{aligned}$$

since $\alpha + \beta = p$, $\alpha\beta = 1$.

2. The condition $(A+B)^2 = A^2 + 2AB + B^2$ is equivalent to

$$AB = BA$$

$$\therefore \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix},$$

i.e.,

$$a - 1 = a + 2b \qquad (1)$$

$$b + 2 = -a + 3b \qquad (2)$$

$$2a + 3 = -3 \qquad (3)$$

$$2b - 6 = -7 \qquad (4)$$

$$(1) \Rightarrow b = -\frac{1}{2} \qquad (5)$$

$$(3) \Rightarrow a = -3 \qquad (6)$$

(5) and (6) satisfy (2) and (4).

3. The intersecting points of the boundary curves

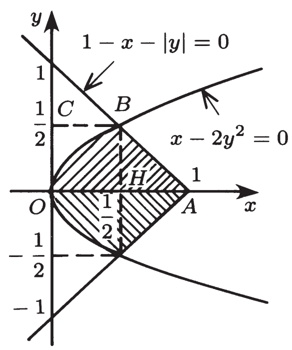
$$x - 2y^2 = 0$$

$$1 - x - |y| = 0$$

are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right).$$

Since the graph is symmetric with respect to the x-axis,



$$S = 2\left[\frac{(\frac{1}{2}+1)\cdot\frac{1}{2}}{2} - \int_0^{1/2} x\,dy\right]$$

$$= \frac{3}{4} - 2\int_0^{1/2} 2y^2\,dy = \frac{7}{12}. \qquad \text{answer}$$

4. First note that $a_n = S_n - S_{n-1}$ for $n \geq 2$.

$$\text{For } n \geq 1, \quad a_1 + 2a_2 + \cdots + na_n = \frac{n+1}{n+2} \qquad (1)$$

$$\text{For } n \geq 2, \quad a_1 + 2a_2 + \cdots + (n-1)a_{n-1} = \frac{n}{n+1} \qquad (2)$$

Subtracting (2) from (1) we have

$$na_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+2)(n+1)}$$

$$a_n = \frac{1}{n(n+1)(n+2)}$$

$$= \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right], \qquad n \geq 2.$$

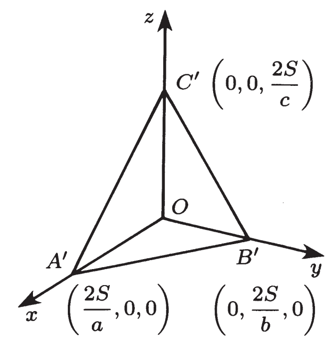
However, $a_1 = \frac{2}{3}$ from (1),

$$\therefore S_n = \frac{2}{3} + \frac{1}{2}\left[\left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right) + \left(\frac{1}{3\cdot 4} - \frac{1}{4\cdot 5}\right) + \cdots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right]$$

$$= \frac{2}{3} + \frac{1}{2}\left[\frac{1}{2\cdot 3} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{3n^2+9n+4}{4(n+1)(n+2)}. \qquad \text{answer}$$

5. (1)



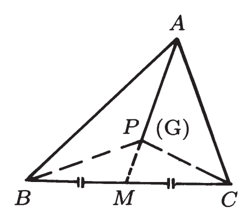
$$\triangle ABC = \triangle PBC + \triangle PCA + \triangle PAB$$

$$= \frac{1}{2}(ax+by+cz). \qquad \text{answer}$$

(2) From (1), point $P'(x, y, z)$ is on the plane

$$ax + by + cz = 2S.$$

Here $x > 0$, $y > 0$, $z > 0$, and P' moves inside of the $\triangle A'B'C'$.



(3) If P is the center of gravity of $\triangle ABC$, the extension of AP meets the midpoint, M, of BC.

$$\triangle ABP = \triangle ACP$$

$$\triangle ABP = \triangle BCP$$

$$\therefore ax = by = cz.$$

By taking this, together with (1), we have

$$x = \frac{2S}{3a}, \qquad y = \frac{2S}{3b}, \qquad z = \frac{2Ss}{3c},$$

which are the coordinates of the center of gravity of $\triangle A'B'C'$.

Examination B
Economics Division

1. When $y = x+k$ is tangent to $y = \sqrt{2x-1}$, there is exactly one point of intersection between them and also exactly one intersecting point between $y = x+k$ and the parabola $y^2 = 2x-1$. Therefore, $2x-1 = (x+k)^2$ has one solution and the discriminant D of $x^2 + 2(k-1)x + k^2 + 1 = 0$ is zero.

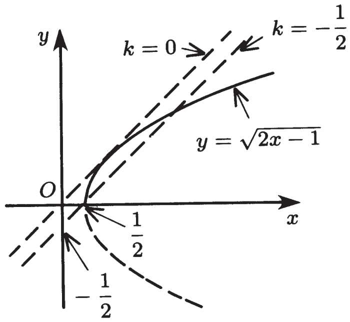
$$(k-1)^2 - (k^2+1) = 0$$

$$\therefore k = 0.$$

If $y = x+k$ passes through the vertex $(\frac{1}{2}, 0)$ of $y = \sqrt{2x-1}$, then

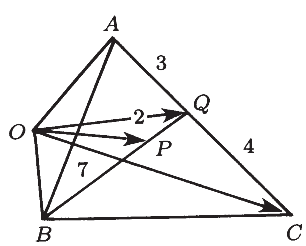
$$k = -\frac{1}{2}.$$

If $k > 0$, there is no point of intersection. If $k = 0$ or $k < -\frac{1}{2}$, there is one point of intersection. If $-\frac{1}{2} \leq k < 0$, there are two points of intersection.



2. (1) $AQ : QC = 3 : 4$ implies that

$$\begin{aligned}\overrightarrow{OQ} &= \frac{4\overrightarrow{OA} + 3\overrightarrow{OC}}{3+4} \\ &= \frac{4}{7}\overrightarrow{OA} + \frac{3}{7}\overrightarrow{OC}. \qquad \text{answer}\end{aligned}$$



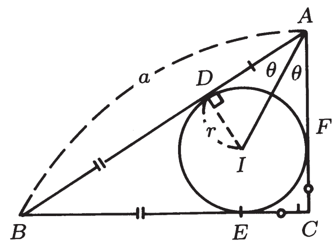
(2)
$$\begin{aligned}\overrightarrow{OP} &= \frac{7\overrightarrow{OQ} + 2\overrightarrow{OB}}{2+7} \\ &= \frac{7(\frac{4}{7}\overrightarrow{OA} + \frac{3}{7}\overrightarrow{OC}) + 2\overrightarrow{OB}}{9} \\ &= \frac{4}{9}\overrightarrow{OA} + \frac{2}{9}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OC}. \qquad \text{answer}\end{aligned}$$

3. (1) Let E, F be points of tangency of the inscribed circle I on side BC, CA, respectively. Then

$$\begin{cases} AD = AF \\ BD = BE \\ CE = CF. \end{cases}$$

Using the above relations

$$\begin{aligned}\frac{1}{2}(AB + AC - BC) &= \frac{1}{2}[(AD + BD) + (AF + CF) \\ &\quad - (BE + CE)] \\ &= AD.\end{aligned}$$



(2) Since $ID \perp AD$, $ID = AD \tan\theta$. Also $AC = AB\cos 2\theta$, $BC = AB \sin 2\theta$.

$$\begin{aligned}\frac{r}{\tan\theta} &= AD \\ &= \frac{1}{2}(a + a\cos 2\theta - a \sin 2\theta) \\ r &= \frac{a}{2}(1 + \cos 2\theta - \sin 2\theta)\tan\theta. \qquad \text{answer}\end{aligned}$$

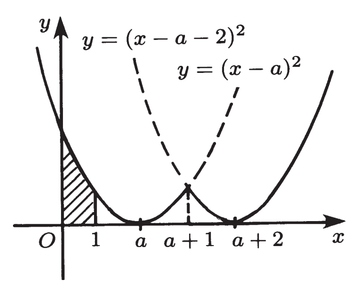
(3) Using the double angle formula for the result of (2), we have

$$\begin{aligned}r &= \frac{a}{2}(2\cos^2\theta - 2\sin\theta\cos\theta) \cdot \frac{\sin\theta}{\cos\theta} \\ &= a(\sin\theta\cos\theta - \sin^2\theta) \\ &= \frac{a}{2}[\sin 2\theta - (1 - \cos 2\theta)] \\ &= \frac{a}{2}\left[\sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right) - 1\right].\end{aligned}$$

$2\theta = \angle A$ is an interior angle of right triangle ABC, so $0 < 2\theta < \pi/2$ and hence $\pi/4 < 2\theta + \pi/4 < \frac{3}{4}\pi$. It follows that the maximum value of r is

$$\frac{a}{2}(\sqrt{2} - 1). \qquad \text{answer}$$

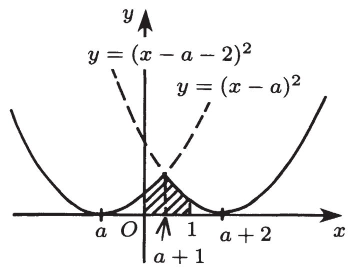
4. The graph of $f(x)$ is shown below with the solid line. At $x = a + 1$, f has a relative maximum.



(i) When $0 \leq a \leq 2$,

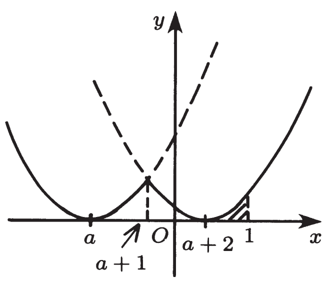
$$\begin{aligned}F(a) &= \int_0^1 (x-a)^2\,dx = \frac{(x-a)^3}{3}\Big|_0^1 \\ &= a^2 - a + \frac{1}{3} = \left(a - \frac{1}{2}\right)^2 + \frac{1}{12}.\end{aligned}$$

(ii) When $-1 \leq a < 0$,



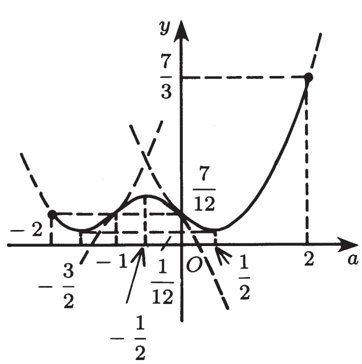
$$F(a) = \int_0^{a+1} (x-a)^2 dx + \int_{a+1}^{1} (x-a-2)^2 dx$$
$$= \left[\frac{(x-a)^3}{3}\right]_0^{a+1} + \left[\frac{(x-a-2)^3}{3}\right]_{a+1}^{1}$$
$$= -a^2 - a + \frac{1}{3}$$
$$= -\left(a+\frac{1}{2}\right)^2 + \frac{7}{12}.$$

(iii) When $-2 \le a < -1$,



$$F(a) = \int_0^1 (x-a-2)^2 dx$$
$$= \frac{(x-a-2)^3}{3}\Big|_0^1$$
$$= a^2 + 3a + \frac{7}{3}$$
$$= \left(a+\frac{3}{2}\right)^2 + \frac{1}{12}.$$

From (i), (ii), (iii) we have the graph of $y = F(a)$ as shown below.



It follows that the maximum value is

$$F(2) = \frac{7}{3}$$

and the minimum value is

$$F\left(\frac{1}{2}\right) = F\left(-\frac{3}{2}\right) = \frac{1}{12}.$$

SOLUTIONS TO SHIGA MEDICAL UNIVERSITY EXAM

Medical School Examination

1. (1) Suppose $m = \sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}$ is a natural number where $a^2 \ge b$ and a, b are natural numbers. Squaring both sides, we have

$$m^2 = 2a + 2\sqrt{a^2-b} \qquad (1)$$

$$\therefore m^2 - 2a = 2\sqrt{a^2-b}. \qquad (2)$$

Squaring both sides, we have

$$(m^2-2a)^2 = 4(a^2-b) \qquad (3)$$

$$\therefore m^4 = 4(am^2-b). \qquad (4)$$

Since the right side is an even number, $m = 2n$ for some natural number n. Then (4) becomes $4n^4 = 4an^2 - b$,

$$\therefore b = 4n^2(a-n^2). \qquad (5)$$

(5) and $b \ge 1$ imply $a - n^2 > 0$, therefore $n^2 < a$. From (3) $(2n)^2 - 2a \ge 0$, therefore $a \le 2n^2$. Together we have

$$n^2 < a \le 2n^2 \quad \text{and} \quad b = 4n^2(a-n^2). \qquad (6)$$

Conversely, if a, b, n satisfy (6), set $m = 2n$ to establish (4) and hence (3). Since $m^2 - 2a \ge 0$, we have (2) and, therefore,

$$m = \sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}.$$

(2) From the conditions in (1)

$$a^2 = 900 \ge b,$$
$$n^2 < 30 \le 2n^2, \qquad (7)$$
$$b = 4n^2(30-n^2).$$

From (7) $n = 4$ or $n = 5$. When $n = 4$, $b = 896 < 900$ and when $n = 5$, $b = 500 < 900$,

$$\therefore b = 500{,}896. \qquad \text{answer}$$

2. (1) From the hypothesis, $bc\sin\theta = 2k$ where k is a constant. Differentiate both sides with respect to t:

$$(c\sin\theta)\frac{db}{dt} + (b\sin\theta)\frac{dc}{dt} + bc\cos\theta\frac{d\theta}{dt} = 0. \qquad (1)$$

θ is an acute angle, so $\cos\theta \neq 0$ and it follows that

$$\frac{d\theta}{dt} = -\frac{\sin\theta}{bc\cos\theta}\left(c\frac{db}{dt} + b\frac{dc}{dt}\right). \qquad (2) \quad answer$$

(2) From the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos\theta$. Differentiate both sides with respect to t:

$$2a\frac{da}{dt} = 2b\frac{db}{dt} + 2c\frac{dc}{dt} - 2\left\{(c\cos\theta)\frac{db}{dt} + (b\cos\theta)\frac{dc}{dt} - (bc\sin\theta)\frac{d\theta}{dt}\right\}.$$

Divide both sides by 2 and substitute (2) in the expression:

$$a\frac{da}{dt} = \left(b - \frac{c}{\cos\theta}\right)\frac{db}{dt} + \left(c - \frac{b}{\cos\theta}\right)\frac{dc}{dt}$$

$$\frac{da}{dt} = \frac{1}{\sqrt{b^2 + c^2 - 2bc\cos\theta}} \times \left\{\left(b - \frac{c}{\cos\theta}\right)\frac{db}{dt} + \left(c - \frac{b}{\cos\theta}\right)\frac{dc}{dt}\right\} \qquad (3) \text{ answer}$$

(3) $$\frac{da}{dt} = \frac{1}{10}\{(10 - 10 \times 2) \times 2 + (10 - 10 \times 2) \times (-1)\} = -1$$ answer

3. (1) An equation for the line AB is

$$\frac{x}{s} + \frac{y}{t} = 1.$$

$$\therefore y = t\left(1 - \frac{x}{s}\right). \qquad (1) \text{ answer}$$

Here $\overrightarrow{AB}^2 = s^2 + t^2 = 1$, so $t = \sqrt{1 - s^2}$; and the y-coordinate of the point of the intersection of (1) and the line $x = x_0$ is

$$y_0 = \sqrt{1 - s^2}\left(1 - \frac{x_0}{s}\right). \qquad (2) \text{ answer}$$

(2) Fix x_0 and vary s between x_0 and 1. We shall investigate the range of y in (1). Let $y = f(s)$ in (2). Then

$$f'(s) = \frac{-s}{\sqrt{1 - s^2}}\left(1 - \frac{x_0}{s}\right) + \sqrt{1 - s^2}\cdot\frac{x_0}{s^2} = -\frac{s^3 - x_0}{s^2\sqrt{1 - s^2}}$$

$$= \frac{(s - \sqrt[3]{x_0})(s^2 + s\sqrt[3]{x_0} + \sqrt[3]{x_0^2})}{s^2\sqrt{1 - s^2}}.$$

Let $\alpha = \sqrt[3]{x_0}$. Since $s^2 + s\alpha + \alpha^2 > 0$, $f'(s) \geq 0 \Leftrightarrow (s - \alpha) \leq 0$.

s	x_0		α		1
$f'(s)$		$+$	0	$-$	
$f(s)$	0	↗	rel. max	↘	0

$$f(\alpha) = \sqrt{1 - \alpha^2}(1 - \alpha^2) = (\sqrt{1 - \alpha^2})^3.$$

That is, $0 \leq y \leq (\sqrt{1 - \alpha^2})^3$. Since $0 \leq x_0 \leq 1$, the range of the region covered by the line AB is

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq \left(\sqrt{1 - \sqrt[3]{x^2}}\right)^3. \qquad \text{answer}$$

4. (1) Let $\overrightarrow{l} = (p, q, r)$.

The normals to Π_1, Π_2 are $\overrightarrow{n_1} = (a, 1, 1)$, $\overrightarrow{n_2} = (1, -a, a)$, respectively. Both are perpendicular to $\overrightarrow{l}$:

$$\begin{cases} \overrightarrow{n_1}\cdot\overrightarrow{l} = ap + q + r = 0 \\ \overrightarrow{n_2}\cdot\overrightarrow{l} = p - aq + ar = 0. \end{cases}$$

$$\therefore p = \frac{-2ar}{a^2 + 1}, \qquad q = \frac{(a^2 - 1)r}{a^2 + 1}.$$

Choose $r = a^2 + 1$. Then

$$\overrightarrow{l} = (-2a, a^2 - 1, a^2 + 1). \qquad \text{answer}$$

(2) Observe that $(-1, a, a)$ is on π_1 and π_2, therefore it is on the line l. An equation of l is

$$\frac{x + 1}{-2a} = \frac{y - a}{a^2 - 1} = \frac{z - a}{a^2 + 1} \qquad (1)$$

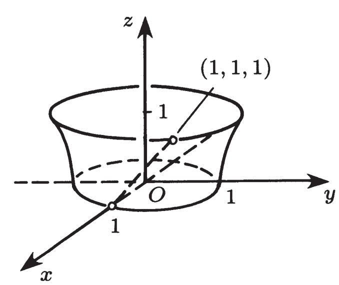
(however, when the denominator is zero, we will set the numerator equals to zero). The x-, y-coordinates of points of intersection of the plane $z = t$ and (1) are obtained by setting $z = t$ in (1):

$$\begin{cases} x + 1 = \dfrac{(t - a)(-2a)}{a^2 + 1} \\ y - a = \dfrac{(t - a)(a^2 - 1)}{a^2 + 1} \end{cases} \quad \therefore \left.\begin{cases} x = \dfrac{-2at + (a^2 - 1)}{a^2 + 1} \\ y = \dfrac{(a^2 - 1)t + 2a}{a^2 + 1} \end{cases}\right\} (2)$$

Eliminating a in (2), we have $x^2 + y^2 = t^2 + 1$, i.e., the curve that results from cutting the locus of l by the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2 + 1}$.

(3) The area of the cross section of this graph cut by $z = t$ is

$$\pi(t^2 + 1).$$



The required volume is

$$\pi\int_0^1 (t^2 + 1)\,dt = \pi\left(\frac{t^3}{3} + t\right)_0^1 = \frac{4\pi}{3}. \qquad \text{answer}$$