NEWS AND LETTERS

74th Annual William Lowell Putnam Mathematical Competition

Editor's Note: Additional solutions will be printed in the *Monthly* later in the year.

PROBLEMS

A1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

A2. Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3,\ 2 \cdot 4,\ 2 \cdot 5,\ 2 \cdot 3 \cdot 4,\ 2 \cdot 3 \cdot 5,\ 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

A3. Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

A4. A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \quad \text{and} \quad N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

A5. For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite for* $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

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A6. Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a, b)$ be as in the table shown; otherwise, let $w(a, b) = 0$.

$w(a,b)$		$b=-2$	-1	0	1	2
a	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. (For example, if $S = \{(0, 1), (0, 2), (2, 0), (3, 1)\}$, then the terms in $A(S)$ are 12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2, -4, -4.)

B1. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n + 1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n + 2).$$

B2. Let $\mathcal{C} = \bigcup_{N=1}^{\infty} \mathcal{C}_N$, where $\mathcal{C}_N$ denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi nx)$$

for which:

(i) $f(x) \geq 0$ for all real x, and
(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through $\mathcal{C}$, and prove that this maximum is attained.

B3. Let $\mathcal{P}$ be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in \mathcal{P}$, then $S \cup S' \in \mathcal{P}$ and $S \cap S' \in \mathcal{P}$, and
(ii) if $S \in \mathcal{P}$ and $S \neq \varnothing$, then there is a subset $T \subset S$ such that $T \in \mathcal{P}$ and T contains exactly one fewer element than S.

Suppose that $f : \mathcal{P} \to \mathbb{R}$ is a function such that $f(\varnothing) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \quad \text{for all } S, S' \in \mathcal{P}.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in \mathcal{P}$?

B4. For any continuous real-valued function f defined on the interval [0, 1], let

$$\mu(f) = \int_0^1 f(x)\,dx, \quad \mathrm{Var}(f) = \int_0^1 \left(f(x) - \mu(f)\right)^2 dx, \quad M(f) = \max_{0 \le x \le 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval [0, 1], then

$$\mathrm{Var}(fg) \le 2\,\mathrm{Var}(f) M(g)^2 + 2\,\mathrm{Var}(g) M(f)^2.$$

B5. Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \ge 0$ such that $f^{(j)}(x) \le k$. [Here $f^{(j)}$ denotes the jth iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f(f^{(j)}(x))$.]

B6. Let $n \ge 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

SOLUTIONS

Solution to A1. Label the vertices $1, 2, \ldots, 12$. Let s_i denote the sum of the numbers on the five faces that meet at vertex i. Because each face contributes its number to exactly three of these vertex sums, we have $\sum_{i=1}^{12} s_i = 3 \cdot 39 < 120$. Thus, the average of the s_i is less than 10, and so there is an i for which $s_i < 10$. However, the sum of any five distinct nonnegative integers is at least $0 + 1 + 2 + 3 + 4 = 10$, so the five numbers that contribute to s_i are not distinct.

Solution to A2. Suppose not. Then there exist m and n in S with $m < n$ and $f(m) = f(n)$; let $n \cdot a_1 \cdot a_2 \cdots a_r$ and $m \cdot b_1 \cdot b_2 \cdots b_s$ be corresponding products as in the problem statement (in particular, they are perfect squares), with a_r, b_s minimal and $a_r = b_s$. If we multiply together these products, put the factors in nondecreasing order, and then omit all factors that appear twice (including $a_r = b_s$), we will have a new product for m as in the problem statement, and the largest remaining factor in this product will be less than b_s, a contradiction.

Solution to A3. The claim is obviously true if $a_0 = a_1 = \cdots = a_n = 0$, so assume this is not the case. Define the nonzero polynomial

$$f(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_n x^{n+1}.$$

If we use the expansion $(1 - x^k)^{-1} = 1 + x^k + x^{2k} + x^{3k} + \cdots$, which is absolutely convergent because $0 < x < 1$, we can rewrite the given condition as

$$f(1) + f(x) + f(x^2) + f(x^3) + \cdots = 0.$$

Because f has finitely many roots, not all the terms in this sum can vanish, so there must be two nonzero terms of opposite sign. If these are $f(x^i)$ and $f(x^j)$ with $i < j$, then by the Intermediate Value Theorem there exists a number y in the interval (x^j, x^i) such that $f(y) = 0$. But then $0 < y < 1$ and $f(y)/y = a_0 + a_1 y + \cdots + a_n y^n = 0$, so we are done.

Solution to A4. Let n be the total number of digits around the circle, and let W denote the whole circle, that is, the unique arc of length n.

Suppose $L > 0$ is any integer. Consider the n arcs of length L beginning at each of the n positions. The total number of zeros they contain is $L\,Z(W)$, and so the average number of zeros they contain is $L\,Z(W)/n$. If this is not an integer, it follows that if w is an arc of length L, then $Z(w)$ is either $\lceil L\,Z(W)/n \rceil$ or $\lfloor L\,Z(W)/n \rfloor$, and both possibilities occur. If $L\,Z(W)/n$ is an integer, it follows that if w is an arc of length L, then $Z(w)$ is either $L\,Z(W)/n$ or one of $(L\,Z(W)/n) \pm 1$; but if one of the latter actually occurred, they couldn't both occur and so the average could not be $L\,Z(W)/n$. Thus $Z(w) = L\,Z(W)/n$ for all such w. Write $\alpha = Z(W)/n$, and note that in either case $Z(w) = \lceil \alpha L \rceil$ or $\lfloor \alpha L \rfloor$ if w has length L, with both possibilities occurring for some w.

Under the assumption in the problem, let $L = N + Z$, so that $L \in \mathbb{Z}$. It is necessary and sufficient to show that there exists an arc of length L with Z zeros. Now consider our family of arcs $w_1, \ldots, w_k$, of lengths $L_1, \ldots, L_k$ respectively, with average length $L \in \mathbb{Z}$ and average number of zeros $Z \in \mathbb{Z}$. Then $Z(w_j) = \lceil \alpha L_j \rceil$ or $Z(w_j) = \lfloor \alpha L_j \rfloor$ for all j. In either case $|Z(w_j) - \alpha L_j| < 1$; thus $|Z - \alpha L| < 1$, and Z is either $\lceil \alpha L \rceil$ or $\lfloor \alpha L \rfloor$. Since both possibilities must occur as $Z(w)$ for some arc w of length L, the result follows.

Solution to A5. Let $T = \triangle DEF$ be an arbitrary triangle in $\mathbb{R}^3$, and let $\mathbf{t} = \overrightarrow{DE} \times \overrightarrow{DF}$; then $\text{Area}(T) = |\mathbf{t}|/2$. If we project T onto a plane P with unit normal vector $\mathbf{n}$, then the area of that projection equals $|\cos\theta| \cdot \text{Area}(T)$, where $0 \le \theta \le \pi$ is the angle between P and the plane that T lies in.

Now think of a variable vector $\mathbf{n}$ ranging over the unit sphere, and the corresponding normal plane P. We can integrate the areas of the projections $\pi_{\mathbf{n}}(T)$ of T onto P over all such $\mathbf{n}$ by using spherical coordinates with $\mathbf{t}$ in the direction of one of the poles. This yields

$$\iint_{S^2} \text{Area}(\pi_{\mathbf{n}}(T))\,dS = \int_0^{\pi}\int_0^{2\pi} |\cos\theta| \cdot \text{Area}(T) \cdot \sin\theta \, d\phi\, d\theta = 2\pi \cdot \text{Area}(T),$$

and, conversely, we can express the area of T using the integral of the areas of the projections.

Now suppose that a given list of numbers a_{ijk} is area definite for $\mathbb{R}^2$. Then, for any points $A_1, \ldots, A_m$ in $\mathbb{R}^3$, we have

$$\begin{aligned}
&\sum_{1 \le i < j < k \le m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \\
&\quad = \frac{1}{2\pi} \sum_{1 \le i < j < k \le m} a_{ijk} \cdot \iint_{S^2} \text{Area}(\pi_{\mathbf{n}}(\triangle A_i A_j A_k))\,dS \\
&\quad = \frac{1}{2\pi} \iint_{S^2} \sum_{1 \le i < j < k \le m} a_{ijk}\,\text{Area}(\triangle \pi_{\mathbf{n}}(A_i)\pi_{\mathbf{n}}(A_j)\pi_{\mathbf{n}}(A_k))\,dS \ge 0,
\end{aligned}$$

where the last step uses the fact that the list of a_{ijk} is area definite for $\mathbb{R}^2$, applied to the points $\pi_{\mathbf{n}}(A_1), \ldots, \pi_{\mathbf{n}}(A_m)$.

Solution to A6. For $0 \le x \le 1, 0 \le y \le 1$ let

$$T(x, y) = \sum_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} w(a, b)\, e^{2\pi i(ax+by)} \quad \text{and} \quad A(x, y) = \sum_{\mathbf{s}=(s_1,s_2)\in S} e^{2\pi i(s_1x+s_2y)}.$$

(Note that $T(x, y)$ is actually a finite sum, by the definition of $w(a, b)$.) Then

$$\begin{aligned} |A(x, y)|^2 &= \sum_{\mathbf{s}\in S} e^{2\pi i(s_1x+s_2y)} \sum_{\mathbf{s}'\in S} e^{-2\pi i(s_1'x+s_2'y)} \\ &= \sum_{(\mathbf{s},\mathbf{s}')\in S\times S} e^{2\pi i((s_1-s_1')x+(s_2-s_2')y)}, \end{aligned}$$

so

$$T(x, y)\, |A(x, y)|^2 = \sum_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} \sum_{(\mathbf{s},\mathbf{s}')\in S\times S} w(a, b)\, e^{2\pi i((a+s_1-s_1')x+(b+s_2-s_2')y)}.$$

If we integrate this over the unit square, each exponential with $a + s_1 - s_1' = b + s_2 - s_2' = 0$ will contribute 1, and all other contributions will be zero. Using the symmetry $w(a, b) = w(-a, -b)$, we can therefore write

$$\int_0^1 \int_0^1 T(x, y)\, |A(x, y)|^2\, dx\, dy = \sum_{(\mathbf{s},\mathbf{s}')\in S\times S} w(\mathbf{s} - \mathbf{s}') = A(S).$$

Thus, to show that $A(S) > 0$, it is enough to show that $T(x, y) \ge 0$ for all x, y and that $T(x, y)\, A(x, y)$ is not identically zero. Grouping the summands in $T(x, y)$ with $(a, b) \ne (0, 0)$ in conjugate pairs, we get

$$\begin{aligned} T(x, y) &= 12 - 8\cos 2\pi x - 8\cos 2\pi y + 4\cos 4\pi x + 4\cos 4\pi y \\ &\quad + 8\cos 2\pi(x+y) + 8\cos 2\pi(x-y) - 4\cos 2\pi(2x+y) - 4\cos 2\pi(2x-y) \\ &\quad - 4\cos 2\pi(x+2y) - 4\cos 2\pi(x-2y) - 2\cos 4\pi(x+y) - 2\cos 4\pi(x-y) \\ &= 12 - 8\cos 2\pi x - 8\cos 2\pi y + 4(2\cos^2 2\pi x - 1) + 4(2\cos^2 2\pi y - 1) \\ &\quad + 16(\cos 2\pi x)(\cos 2\pi y) - 8(2\cos^2 2\pi x - 1)\cos 2\pi y \\ &\quad - 8(2\cos^2 2\pi y - 1)\cos 2\pi x - 4(2\cos^2 2\pi x - 1)(2\cos^2 2\pi y - 1) \\ &= 16(a^2 + b^2 + ab - a^2b - ab^2 - a^2b^2), \end{aligned}$$

where $a = \cos 2\pi x$ and $b = \cos 2\pi y$. Now $-1 \le a, b \le 1$; if also $ab \ge 0$, then

$$a^2 + b^2 + ab - a^2b - ab^2 - a^2b^2 = a^2(1 - b) + b^2(1 - a) + ab(1 - ab) \ge 0.$$

On the other hand, if $ab \le 0$, then

$$a^2 + b^2 + ab - a^2b - ab^2 - a^2b^2 = (a + b)^2 - ab(1 + a)(1 + b) \ge 0.$$

So we have shown that $T(x, y) \ge 0$ for all x, y. Also, we see from the above that $T(x, y)$ can only be zero if a and b are in the set $\{-1, 0, 1\}$, that is, if x and y are integer multiples of $1/4$. On the other hand, $A(0, 0)$ is the cardinality of S, so by continuity, $A(x, y)$ is nonzero for x, y sufficiently close to zero. Hence $T(x, y)\, A(x, y)$ is not identically zero, and we are done.

Solution to B1. From the rules, we have $c(2m)c(2m+2) = c(m)c(m+1)$ and

$$c(2m+1)c(2m+3) = (-1)^m c(m)(-1)^{m+1}c(m+1) = -c(m)c(m+1).$$

Therefore, the terms in the sum cancel in pairs, starting with the term for $n = 2$. Thus the sum is equal to $c(1)c(3) = -1$.

Solution to B2. First note that $\cos(2\pi n/3) = -1/2$ for all integers n not divisible by 3. Therefore, for any f in $\mathcal{C}_N$,

$$f(1/3) = 1 - \frac{1}{2}\sum_{n=1}^{N} a_n \geq 0,$$

and so

$$\sum_{n=1}^{N} a_n \leq 2 \quad \text{and} \quad f(0) = 1 + \sum_{n=1}^{N} a_n \leq 3.$$

To show that 3 is in fact attained as a maximum value, consider

$$f(x) = 1 + \frac{4}{3}\cos(2\pi x) + \frac{2}{3}\cos(4\pi x),$$

which clearly satisfies (ii). As for (i), this can be checked by finding the critical points of f, or by noting that

$$f(x) = \frac{1}{3}(2\cos(2\pi x) + 1)^2 = \frac{1}{3}\left(\frac{\sin(3\pi x)}{\sin(\pi x)}\right)^2.$$

Solution to B3. Yes. First we show that if $T_1, T_2 \in \mathcal{P}$, $i \notin T_1, T_2$, and $T_1 \cup \{i\}$, $T_2 \cup \{i\} \in \mathcal{P}$, then

$$f(T_1 \cup \{i\}) - f(T_1) = f(T_2 \cup \{i\}) - f(T_2).$$

To see this, apply the condition on f to $S = T_1 \cup \{i\}$, $S' = T_2$ to get

$$f(T_1 \cup T_2 \cup \{i\}) = f(T_1 \cup \{i\}) + f(T_2) - f(T_1 \cap T_2),$$

and also apply it to $S = T_1$, $S' = T_2 \cup \{i\}$ to get

$$f(T_1 \cup T_2 \cup \{i\}) = f(T_1) + f(T_2 \cup \{i\}) - f(T_1 \cap T_2);$$

subtract the two equations from each other. Now we can define f_i to be the value of $f(T \cup \{i\}) - f(T)$ for any set $T \in \mathcal{P}$ for which $i \notin T$ and $T \cup \{i\} \in \mathcal{P}$; if there is no such set T, f_i can be chosen arbitrarily. Then the desired equation

$$f(S) = \sum_{i \in S} f_i$$

follows by induction on the cardinality of S, using (ii).

Solution to B4. For any f, the quantity

$$\int_0^1 \left(f(x) - a\right)^2 dx = \int_0^1 (f(x))^2\, dx - 2\mu(f)a + a^2$$

is a quadratic polynomial in a, which has its minimum when $a = \mu(f)$. Applying this to fg, we have

$$\operatorname{Var}(fg) = \int_0^1 \big(f(x)g(x) - \mu(fg)\big)^2\,dx \le \int_0^1 \big(f(x)g(x) - \mu(f)\mu(g)\big)^2\,dx.$$

Now note that

$$f(x)g(x) - \mu(f)\mu(g) = (f(x) - \mu(f))g(x) + (g(x) - \mu(g))\mu(f),$$

and that

$$\begin{aligned}&\big((f(x) - \mu(f))g(x) + (g(x) - \mu(g))\mu(f)\big)^2\\ &\qquad \le 2\Big(\big((f(x) - \mu(f))g(x)\big)^2 + \big((g(x) - \mu(g))\mu(f)\big)^2\Big).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^1 \big(f(x)g(x) - \mu(f)\mu(g)\big)^2\,dx\\ &\qquad \le 2\int_0^1 \big((f(x) - \mu(f))g(x)\big)^2\,dx + 2\int_0^1 \big((g(x) - \mu(g))\mu(f)\big)^2\,dx\\ &\qquad \le 2\int_0^1 \big(f(x) - \mu(f)\big)^2\,dx\,M(g)^2 + 2\int_0^1 \big(g(x) - \mu(g)\big)^2\,dx\,M(f)^2,\end{aligned}$$

proving the stated result.

Solution to B5 (based on a student paper). First note that to each of the functions to be counted we can associate a rooted forest (disjoint union of trees) with k connected components (the trees) with roots labeled $1, 2, \ldots, k$ and vertices labeled $1, 2, \ldots, n$, namely the graph F with vertex set X and edges from x to $f(x)$ whenever $x \in X$ and $x > k$. The condition on the functions f is precisely what is needed to ensure that every vertex x does occur in exactly one of the trees (the one with root $f^{(j)}(x)$, where $j \ge 0$ is minimal such that $f^{(j)}(x) \le k$) and that there are no cycles. In fact, there is a bijection between the set of functions to be counted and the set of ordered pairs (F, g), where F is such a forest and g is any function from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$. In the forward direction, given f, the forest F is defined as above and the function g is the restriction of f to $\{1, 2, \ldots, k\}$. In the reverse direction, given (F, g), the function f is defined by $f(x) = g(x)$ for $x \in \{1, 2, \ldots, k\}$ (that is, when x is one of the roots) and $f(x) =$ the parent of x for $x > k$.

The number of possible functions g is n^k, so it remains to count the number of possible rooted forests F. Note that because the number of edges of any tree is one fewer than the number of vertices, the number of edges of such a forest is $n - k$. We now temporarily drop the restriction that the set of roots be $1, 2, \ldots, k$ and, on the other hand, add a labeling on the *edges* of the forest.

Claim: The number of rooted forests with k connected components, vertices labeled $1, 2, \ldots, n$, and edges labeled $1, 2, \ldots, n - k$ is

$$A(n,k) = n^{n-k}(n-1)(n-2)\cdots(k+1)k = n^{n-k}\binom{n-1}{k-1}\cdot(n-k)!\,.$$

The proof of the claim is by induction on $n - k$; for the base case $n - k = 0$, there is just one possible forest, and all is well. Given a rooted forest with $k + 1$ components, vertices labeled $1, 2, \ldots, n$, and edges labeled $1, 2, \ldots, n - k - 1$, we can make a rooted forest with one fewer component by choosing any vertex v of a tree and connecting it to one of the k roots of the *other* trees by a new edge labeled $n-k$, thus creating a "merged" tree (with the same root as the original tree containing v). This process will create every rooted forest with k connected components, vertices labeled $1, 2, \ldots, n$, and edges labeled $1, 2, \ldots, n-k$ exactly once; because there are n choices for the vertex v and k choices for the root to connect it to, we have

$$A(n,k) = A(n,k+1) \cdot n \cdot k\,,$$

and the claim follows.

Having established the claim, we discard the labeling on the edges, which results in the number of forests being divided by $(n-k)!$, and we reintroduce the restriction that the roots be $1, 2, \ldots, k$, which results in a further division by $\binom{n}{k}$ (the number of possible sets of k roots, only one of which is acceptable). Therefore, the number of possible rooted forests F above is

$$\frac{A(n,k)}{(n-k)!\binom{n}{k}} = n^{n-k-1}k,$$

so the number of ordered pairs (F, g) is

$$n^k \cdot n^{n-k-1}k = k \cdot n^{n-1},$$

and we are done.

Solution to B6. Write $n = 2k-1$. Alice must play in space k (the middle space).

We claim that a position in which fewer than $n-1$ spaces are occupied is a winning position for the current player if and only if the sum of the occupied spaces is not an odd multiple of k.

Note that the number of stones cannot decrease unless all spaces are occupied. Moreover, suppose there are at least two empty spaces. Then at each move, the number of stones increases by one unless a player removes a stone from a sequence of stones of the form $1, \ldots, j$ or $j, \ldots, n$; and in that case the sum $\min\{i : i \text{ empty}\} + (n - \max\{i : i \text{ empty}\})$ always decreases. It follows that no sequence of moves could cause a position to be repeated (even if that were permitted), except for the positions with at least $n-1$ stones.

Observe that any position with at least $n-1$ stones, in other words with 0 or 1 empty spaces, can be turned into any other (different) such position with a single move. It follows that once there are $n-1$ stones on the board, there are exactly n moves left in the game; since n is odd, the player with the first opportunity to play in that situation will win. Note that if the only unoccupied space is space i, then the sum of the occupied spaces is $k(2k-1) - i$, which is not an odd multiple of k.

Finally, note that in any position with at least two empty spaces, the $2k-1$ available moves can change the sum of the occupied spaces by any of $1, \ldots, 2k-1 \pmod{2k}$. First, playing in an unoccupied space i changes the sum by i. Second, removing a stone from space $j < i < J$ with j, J unoccupied but $j+1, \ldots, J-1$ occupied (allowing $j = 0$ and/or $J = 2k = n+1$) changes the sum by $j + J - i$; as i ranges from $j+1$ to $J-1$, the change in sum ranges from $J-1$ to $j+1$. So to change the sum by $i \pmod{2k}$, put a stone in space i if it is unoccupied; otherwise, find j, J as above and remove the stone from space $J + j - i$.

Thus, given a position with at least two empty spaces, if the sum is initially not an odd multiple of k, it can always be made so; while if the sum is initially an odd multiple of k, it can never remain so. The claim follows.