Prime Numbers: Progress and Pitfalls

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c) The Riemann Hypothesis - Also includes a $ 1,000,000 reward.

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Often for teaching calculus or math service courses.

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For the Amateur interested in studying primes without expecting proofs or fame:

1. They can have fun and provide useful ideas, computations, questions, conjectures

2. But they have a hard time publishing their work.

1. Primes often exhibit their properties beyond any doubt empirically but we have no way to prove these properies hold.

Why do Primes Make Fools Out of All of Us?

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1. Primes Often Do Not Hide Their True Nature

The 25 prime numbers less than 100:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67,$
$71, 73, 79, 83, 89, 97$

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The first 10 primes starting at a billion:

1000000007, 1000000009, 1000000021, 1000000033, 1000000087, 1000000093, 1000000097, 1000000103, 1000000123, 1000000181

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You never run out because if $p_1, p_2, p_3, \ldots p_k$ are k primes, then let

$$N = P + 1, \quad P = p_1 p_2 p_3 \cdots p_k$$

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and N will have all new prime factors since

$$\frac{N}{p_1} = p_2 p_3 \cdots p_k + \frac{1}{p_1} \quad \text{is not an integer.}$$
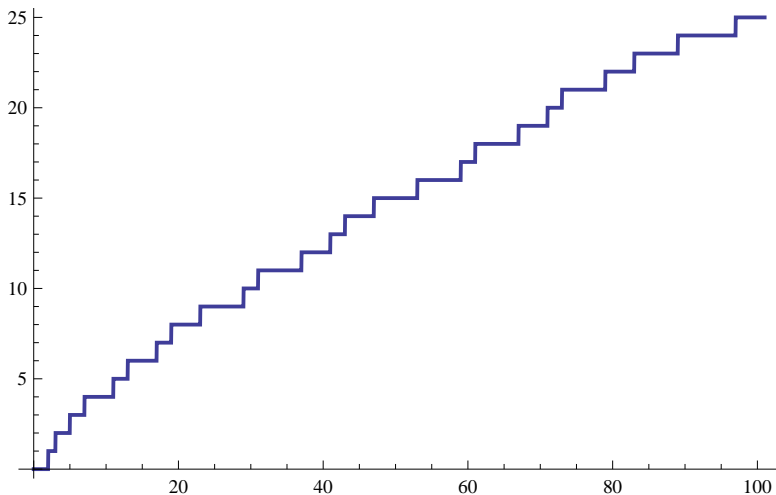
Let
$$\pi(x) = \text{ the number of primes} \leq x = \sum_{p \leq x} 1,$$

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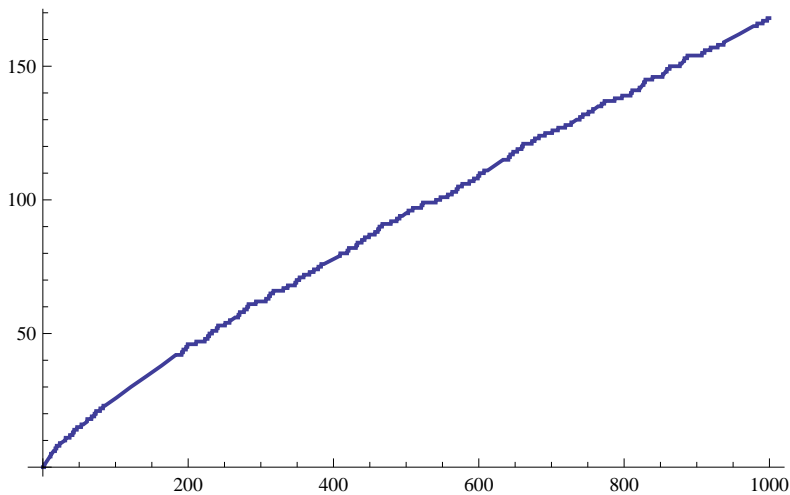Figure: $\pi(x)$, the number of primes p for $2 \leq x \leq 100$
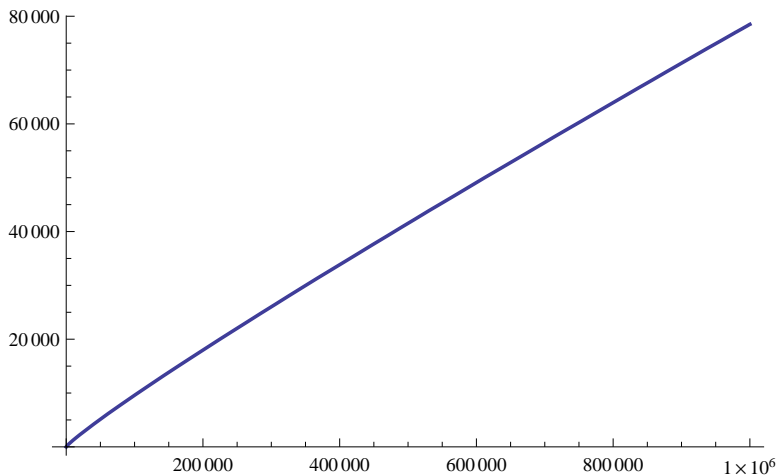
Figure: $\pi(x)$ for $0 \le x \le 1000$

Figure: $\pi(x)$ for $0 \leq x \leq 10^6$

This suggests the Prime Number Theorem:

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{i. e.} \quad \lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1, \quad \log x = \log_e x = \ln x.$$

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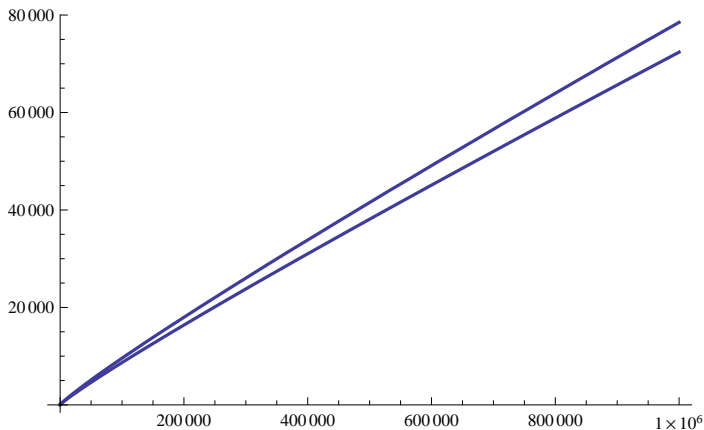Figure: $\pi(x)$ and $\frac{x}{\log x}$ for $1 \le x \le 10^6$

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Probability(n is a prime) $= \frac{1}{\log n}$

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Probability(n is a prime) $= \frac{1}{\log n}$

Thus

$$\pi(x) \sim \sum_{n \leq x} \frac{1}{\log n} \sim \text{li}(x),$$

where

$$\text{li}(x) = \int_2^x \frac{1}{\log t} \, dt$$
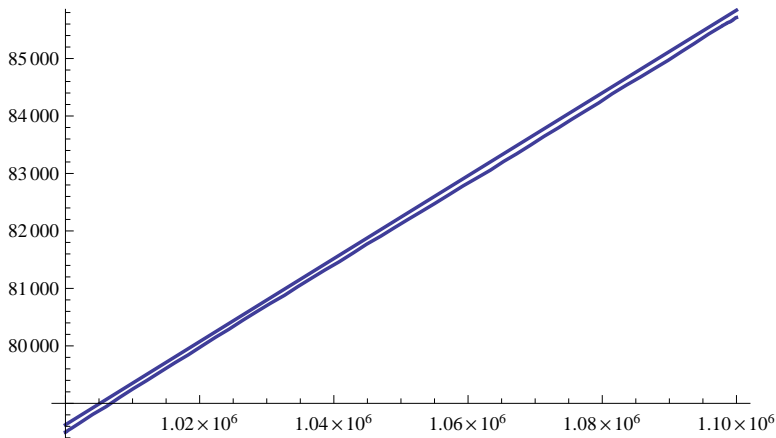
is called the logarithmic integral.

Figure: $\pi(x)$ and $\mathrm{li}(x)$ for $10^6 \leq x \leq 10^6 + 10^5$

Twin Primes

Now consider the sequence

$$3, 5, 7, 11, 13, 17, 19, 29, 31, 41, 43, 59, 61, 71, 73, 101, 103, \ldots .$$

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$$3, 5, 7, 11, 13, 17, 19, 29, 31, 41, 43, 59, 61, 71, 73, 101, 103, \ldots .$$

These are pairs of primes that are two apart: the sequence of twin primes.

Here are the twin primes starting at a billion:

1000000007, 1000000009, 1000000409, 1000000411, 1000000931, 1000000933, 1000001447, 1000001449, 1000001789, 1000001791, 1000001801, 1000001803.

Let

$$\pi_2(x) = \text{the number of pairs of twin primes } \leq x.$$

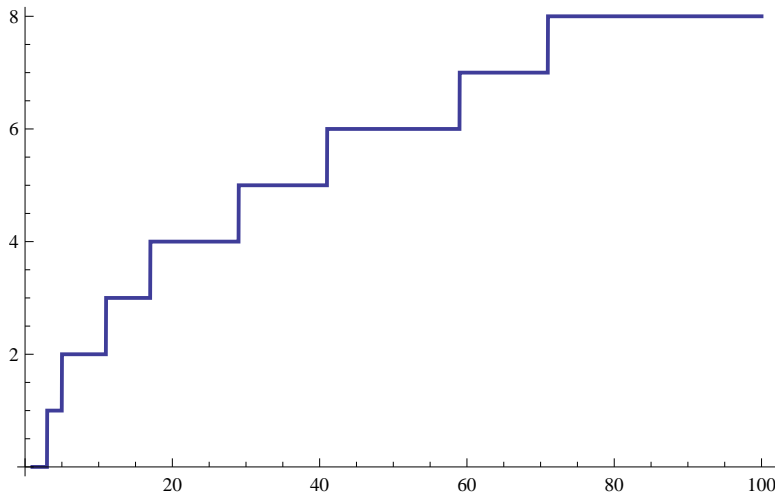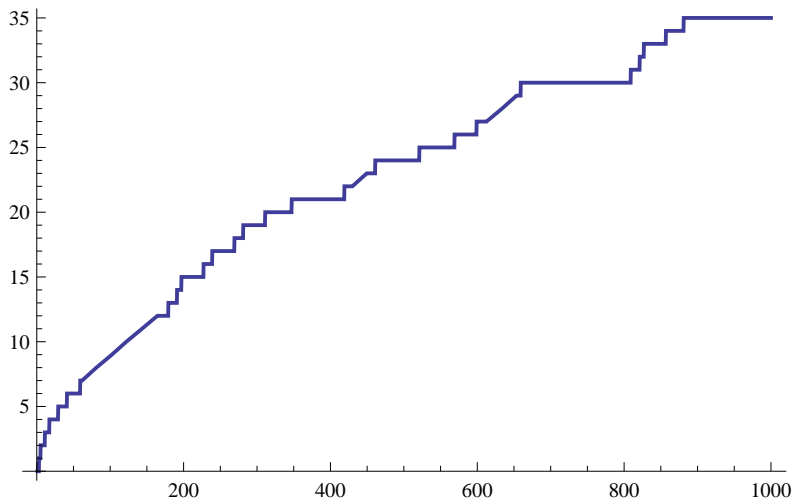Figure: $\pi_2(x)$ for $0 \leq x \leq 100$

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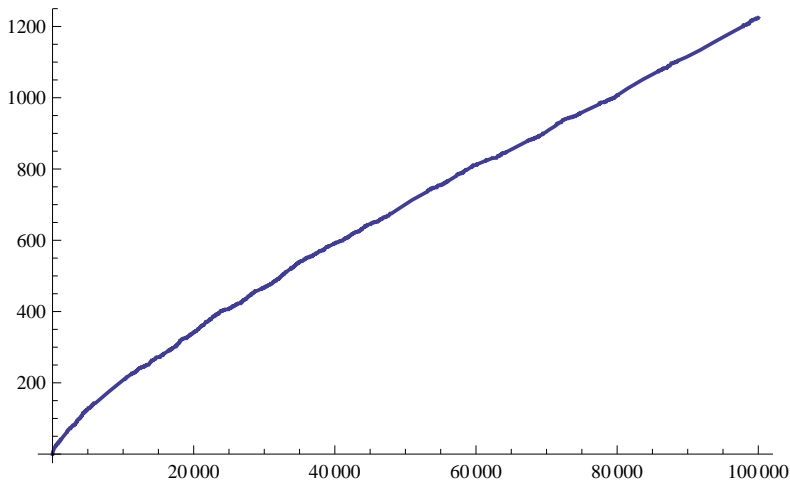
Figure: $\pi_2(x)$ for $0 \le x \le 10^5$

As for the Prime Number Theorem, we conjecture

$$\pi_2(x) \sim (1.3203\ldots)\int_2^x \frac{dt}{(\log t)^2}.$$
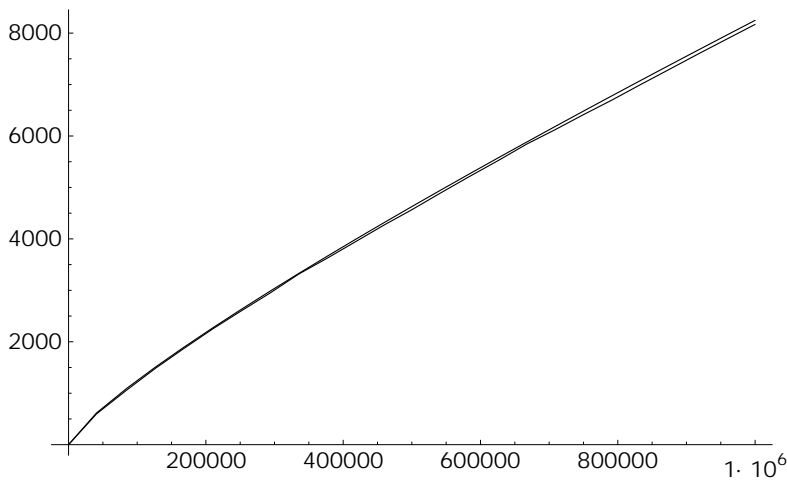
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The chance that n is prime is $1/\log n$.

The chance that $n + 2$ is prime is $1/\log(n + 2) \sim 1/\log n$.

Therefore by independence the probability of both being prime is $1/(\log n)^2$. Thus

$$\pi_2(x) \sim \int_2^x \frac{1}{(\log t)^2} \, dt$$

The problem with the Cramér's model is that it fails to take into account divisibility. Thus, for the primes $p > 2$, the probability that $p + 1$ is prime is not $1/\log(p+1)$ as suggested by the Cramér model but rather 0 since $p + 1$ is even.

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Further $p + 2$ is necessarily odd; therefore it is twice as likely to be prime as a random number. The conclusion is that n and $n + 2$ being primes are not independent events.

Correction: We need both n and $n + 2$ to not be divisible by $2, 3, 5, 7, 11, \cdots$.

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The chance that two random numbers are both odd is $(1/2)(1/2) = 1/4$, but, since n being odd forces $n + 2$ to be odd, the chance that n and $n + 2$ are both odd is $1/2$, and thus twice as large as random.

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The chance that two random numbers are both not divisible by 3 is $(2/3)(2/3) = 4/9$, but the chance that n and $n+2$ are not both divisible by 3 is $1/3$ since this occurs if and only if n is congruent to 2 modulo 3.

In general, the probability that two random numbers are not divisible by $p > 2$ is $(1 - 1/p)^2$, while the probability that both n and $n + 2$ are not divisible by p is the slightly smaller $1 - 2/p$ since n must miss the two residue classes 0 and -2 modulo p.

In general, the probability that two random numbers are not divisible by $p > 2$ is $(1 - 1/p)^2$, while the probability that both n and $n + 2$ are not divisible by p is the slightly smaller $1 - 2/p$ since n must miss the two residue classes 0 and -2 modulo p.

Therefore, the correction factor to the Cramér model for lack of independence is 2 if $p = 2$, and for $p \geq 3$ is

$$\left(1 - \frac{2}{p}\right)\left(1 - \frac{1}{p}\right)^{-2} = \left(\left(1 - \frac{1}{p}\right)^2 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p}\right)^{-2}$$
$$= \left(1 - \frac{1}{(p-1)^2}\right).$$

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$$= \left(1 - \frac{1}{(p-1)^2}\right).$$

We conclude that the correct approximation for twin primes should be

$$\pi_2(x) \sim 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\frac{x}{(\log x)^2}.$$

Let $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ be a set of k distinct integers, and denote by $\pi(x; \mathcal{H})$ the number of positive integers $n \leq x$ for which $n + h_1, n + h_2, \ldots, n + h_k$ are all primes.

Hardy-Littlewood Conjecture

Let $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ be a set of k distinct integers, and denote by $\pi(x; \mathcal{H})$ the number of positive integers $n \leq x$ for which $n + h_1, n + h_2, \ldots, n + h_k$ are all primes.

In 1923, Hardy and Littlewood conjectured for $x \to \infty$

$$\pi(x; \mathcal{H}) \sim \mathfrak{S}(\mathcal{H}) \mathrm{li}_k(x),$$

where

$$\mathrm{li}_k(x) = \int_2^x \frac{dt}{(\log t)^k},$$

and $\mathfrak{S}(\mathcal{H})$ is the singular series defined by the product over all primes p

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right),$$

where $\nu_{\mathcal{H}}(p)$ is the number of distinct residue classes occupied by the elements of $\mathcal{H}$.

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$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right),$$

where $\nu_{\mathcal{H}}(p)$ is the number of distinct residue classes occupied by the elements of $\mathcal{H}$. We assume $\mathfrak{S}(\mathcal{H}) \neq 0$ here.

The Jumping Champion Problem

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> *"Find the most probable difference between consecutive primes."*

Editor's Comment in the 1978-1979 volume:

> *"No solution has been received, though there has been a good deal of evidence presented pointing to the reasonable conjecture that there is no most probable difference between consecutive primes. On the other hand, there is also some evidence that 6 is the most probable such difference . . . However, there seems to be good reason to expect that 30 will eventually replace 6 as the most probable difference and still later 210, 2310, 30030, etc. will have their day."*

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Odlyzko started talking about the problem in the early 90's. In 1993 J. H. Conway invented the term jumping champion to refer to the most frequently occurring difference between consecutive primes less than or equal to x.

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$$N(x, d) := \sum_{\substack{p_n \leq x \\ p_n - p_{n-1} = d}} 1$$

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$$N^*(x) := \max_d N(x, d).$$

The set $D^*(x)$ of jumping champions for primes $\leq x$ is given by

$$D^*(x) := \{d^* \colon N(x, d^*) = N^*(x)\}.$$

Example

$x = 100$

Primes $\leq x$

>2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59,
>61, 67, 71, 73, 79, 83, 89, 97

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Difference:

> $1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8$

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$$N(100, 1) = 1, N(100, 2) = 8, N(100, 4) = 7,$$
$$N(100, 6) = 7, N(100, 8) = 1.$$

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$$N(100, 1) = 1, N(100, 2) = 8, N(100, 4) = 7,$$
$$N(100, 6) = 7, N(100, 8) = 1.$$

$$N^*(100) = 8, \quad d^* = 2$$

Table 1 below summarizes everything we presently know about jumping champions less than or equal to x.

Table: Known jumping champions for small x.

$D^*(x)$	Smallest Prime Occurrence of x	Largest Known Prime Occurrence of x
$\{1\}$	3	3
$\{1, 2\}$	5	5
$\{2\}$	7	433
$\{2, 4\}$	101	173
$\{4\}$	131	541
$\{2, 4, 6\}$	179	487
$\{2, 6\}$	379	463
$\{6\}$	389	$> 10^{15}$
$\{4, 6\}$	547	941

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Their concern was finding the transition between 6 and 30, which they estimated to be at $1.7427 \cdot 10^{35}$. (30 transitions to 210 at about 10^{425} ?)

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However nothing has actually been proved about Jumping Champions. For all we know 2 is the jumping champion for all large x instead of the biggest loser.

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The Polymath 8b project now had obtained obtained 246.

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The goal of this method is to show there are two or more primes in the tuple

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for infinitely many n, for some $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ This is equivalent to showing

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Let

$$a_n := \frac{1}{(k+\ell)!} \sum_{\substack{d \mid (n+h_1) \cdots (n+h_k) \\ d \leq R}} \mu(d)(\log(R/d))^{k+\ell}$$

Then a_n is usually very small except when $(n + h_1) \cdots (n + h_k)$ has few prime factors.

For $R < N^{1/4-\epsilon}$ we can work out asymptotically both

$$S := \sum_{n \leq N} (a_n)^2$$

and

$$S_i(\mathcal{P}) := \sum_{n \leq N} 1_{\mathcal{P}}(n + h_i)(a_n)^2$$

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Conditionally if primes are well distributed in arithmetic progressions, then we can take $R < N^{1/2-\epsilon}$. This then gives that for $k = 7$

$$S_i(\mathcal{P}) > \frac{1}{k} S$$

which proves there are two primes closer than 20 apart infinitely often. Unconditionally with $R = N^{1/4-\epsilon}$ you get two primes closer than $\epsilon \log p_n$:

$$\liminf_{n \to \infty} \left(\frac{p_{n+1} - p_n}{\log p_n} \right) = 0$$

The GPY Method in a Nutshell

We consider the weight function

$$\Lambda_R(n, \mathcal{H}_k, \ell) = \frac{1}{(k+\ell)!} \sum_{\substack{d \mid (n+h_1)(n+h_2)\cdots(n+h_k) \\ d \leq R}} \mu(d) \log(R/d)^{k+\ell}$$

Here the parameter ℓ gives needed freedom. This weight is large for n where the numbers in the set or tuple $\mathcal{H}_k = \{n + h_1, n + h_2, \ldots, n + h_k\}$ are primes or only have a few prime factors, and is small otherwise.

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More generally

$$\Lambda_R(n, \mathcal{H}_k, P) = \sum_{\substack{d | (n+h_1)(n+h_2)\cdots(n+h_k) \\ d \leq R}} \mu(d) P(\log(R/d)),$$

where $P(x) = \sum_{j \geq k} a_j x^j$.

To detect primes among the numbers in
$\mathcal{H}_k = \{n + h_1, n + h_2, \ldots, n + h_k\}$ for $N < n \le 2N$ consider

$$S(\mathcal{H}_k, m) = \sum_{n=N+1}^{2N} \left(\sum_{i=1}^{k} 1_p(n + h_i) - m \right) \Lambda(n, \mathcal{H}_k, P)^2.$$

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This is the GPY method, together with several different techniques for evaluating $S(m)$ asymptotically when $R < N^{\frac{1}{4}}$ (or conditional $R < N^{\frac{1}{2}-\epsilon}$ on the Elliott-Halberstam Conjecture.

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2. You have to work on bounded gaps between primes, not give up working on bounded gaps between primes.

What was Mathematically needed for the Breakthroughs in 2013-14

The GPY method never worked unconditionally to find Bounded Gaps between Primes. Zhang showed that one could restrict the GPY method to divisors without large prime divisors, and then improve the level of distribution results for primes in arithmetic progressions with modulus these numbers.

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In late 2013 and 2014 James Maynard and independently Terry Tao found that using more general weights of the form

$$\Lambda_R(n, \mathcal{H}_k, F) = \sum_{\substack{d_i \mid (n+h_i), 1 \le i \le k \\ d_1 d_2 \ldots d_k \le R}} \mu(d_1)\mu(d_2)\cdots\mu(d_k)\Lambda_{d_1, d_2, \ldots, d_k}$$

where the Λ's are complicated smooth weights in the GPY Method proves not just Bounded Gaps Between Primes exist, but also Bound Gaps between any fixed number of primes!

The Sieve of Eratosthenes: It's Bad for Proving Anything

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 2$

Get prime 2, remove multiples of 2

$1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29$

Get prime 3, remove multiples of 3

$1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 25, 29$

Get prime 5, remove multiples

$1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29$

How much is left by the sieve?

$$\lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor \dots$$

The proportion is actually

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \dots$$

The Sieve of Eratosthenes: It's Bad for Proving Anything

Here is how we do this in equations: Let $\mu(1) = 1$, $\mu(p) = -1$, $\mu(p^2) = 0$ and extend $\mu(n)$ multiplicatively, so that $\mu(p_1 p_2 p_3 \cdots p_r) = (-1)^r$. Then you can check that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

Now, let $P(y) = \prod_{p \leq y} p$, Then

$$\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{\substack{n \leq x \\ (n, P(\sqrt{x})) = 1}} 1.$$

Next

$$\sum_{\substack{n \leq x \\ (n, P(\sqrt{x})) = 1}} 1 = \sum_{n \leq x} \left(\sum_{d | (n, P(\sqrt{x}))} \mu(d) \right)$$

Since $d|(n, P(\sqrt{x}))$ is equivalent to $d|n$ and $d|P(\sqrt{x})$, this is

$$= \sum_{d|P(\sqrt{x})} \mu(d) \sum_{\substack{n \leq x \\ d|n}} 1 = \sum_{d|P(\sqrt{x})} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$= x \sum_{d|P(\sqrt{x})} \frac{\mu(d)}{d} - \sum_{d|P(\sqrt{x})} \mu(d) \left\{ \frac{x}{d} \right\}$$

$$= x \prod_{p \leq \sqrt{x}} \left(1 - \frac{1}{p} \right) - \sum_{d|P(\sqrt{x})} \mu(d) \left\{ \frac{x}{d} \right\}.$$

By Merten's theorem the first term is $\sim 2e^{-\gamma} \frac{x}{\log x}$, Hence

$$\pi(x) - \pi(\sqrt{x}) + 1 \sim 2e^{-\gamma} \frac{x}{\log x} - \sum_{d|P(\sqrt{x})} \mu(d) \left\{ \frac{x}{d} \right\}.$$

Thus we get prime number theorem wrong!

Modern Sieve Methods

The number of $d|P(\sqrt{x})$ is $\sim e^{2\sqrt{x}/\log x}$ which is huge. To keep this from swamping the main term we would need to replace $\sqrt{x}$ by $\leq c(\log x)$ which is tiny.

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Solution: Try to get upper and lower bounds only:

$$\sum_{d|n} \alpha_d \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \beta_d$$

Brun chose α_d and β_d to be $\mu(d)$ or 0 depending on if d has $\leq r$ distinct prime factors, and whether r is odd or even. In 1947 Selberg made a brillant choose for β_d : Let $\lambda_1 = 1$, λ_d arbitrary real, then

$$\sum_{d|n} \mu(d) \leq \left(\sum_{d|n} \lambda_d \right)^2.$$

Now choose $\lambda_d = 0$ for $d > z$, and optimize.