Rubik's Tesseract

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The great popularity of Ernö Rubik's ingenious cubical puzzle led to the appearance of many variations on Rubik's idea: a $4 \times 4 \times 4$ cube, puzzles in the shape of tetrahedra and dodecahedra, etc. One natural variation that never appeared on toy store shelves is the four-dimensional version of Rubik's cube—what might be called a "Rubik's tesseract." In this paper we consider the mathematics of the $3 \times 3 \times 3 \times 3$ Rubik's tesseract. This has also been studied independently by H. R. Kamack and T. R. Keane (see [2]), Joe Buhler, Brad Jackson, and Dave Sibley.

Of course, the tesseract is somewhat harder to work with than the cube, since we can't build a physical model and experiment with it. The results described below were discovered with the aid of a simulation of the tesseract on a Macintosh computer. In this simulation the computer displays a representation of the tesseract on the screen, and the user uses a pointing device (a mouse) to ask the computer to twist sides of the tesseract. To understand the graphic representation of the tesseract used in this simulation, it might be helpful to consider first the easier problem of representing the ordinary three-dimensional Rubik's cube in a way that two-dimensional people could understand.

One way to make a two-dimensional representation of the Rubik's cube would be to imagine unfolding the surface of the cube in the familiar way illustrated in FIGURE 1. (In all of the figures in this paper, the colors of the sides of the cube are represented as black-and-white patterns. We will continue to refer to them, however, as colors.) Unfortunately, this representation would not be very useful to a two-dimensional person trying to solve the cube. The problem is that this representation doesn't show clearly which colors are attached to different sides of the same cubie. (The 27 small cubes that make up a Rubik's cube are usually called "cubies.") For example, the three "colors" attached to the cubie in the front bottom right corner of the cube in FIGURE 1 are stripes, gray, and dots, but it takes some thought to figure this out from the two-dimensional representation in FIGURE 1.

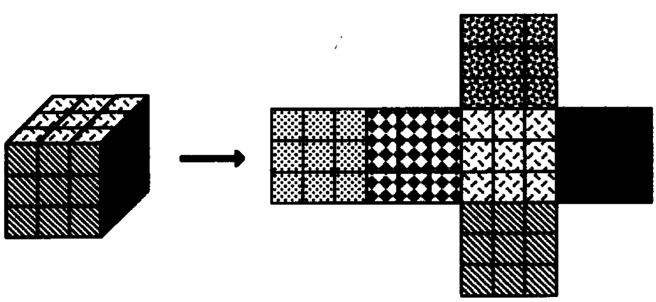

FIGURE 1

Here's a more useful way to make a two-dimensional representation of the cube. Imagine slicing the cube horizontally into three layers, and then spreading these layers out and viewing them from above (see FIGURE 2). Each layer would look like a 3×3 square, with the same four colors appearing along the sides of all three layers.

In addition, two more colors appear in the interiors of all the squares in the first and third layers; these are the colors that face down and up on the cube. (You have to imagine that you can see through the cubies in the lower layer, to see from above the dots on their bottom faces.) A two-dimensional person viewing this picture would not be able to visualize how the three square layers should be stacked up vertically to form a cube, or the directions in which the colors in the interiors of the first and third layers face on this cube. However, this representation has the advantage that the 27 small squares in this two-dimensional picture correspond to the 27 cubies of the Rubik's cube, and the colors attached to each cubie on the Rubik's cube are also attached to the corresponding square in the two-dimensional representation.

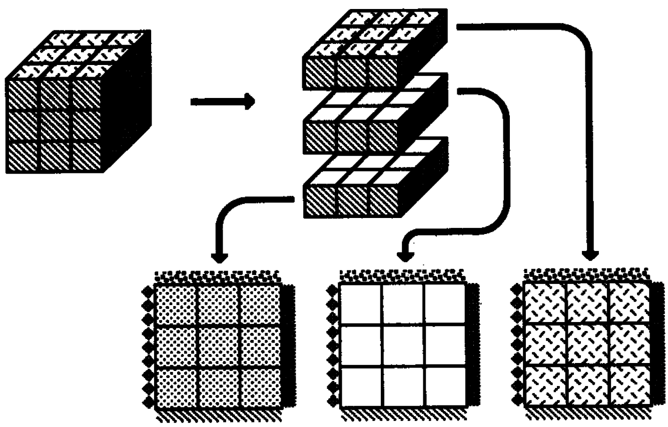

FIGURE 2

Clearly, twisting the bottom or top of the Rubik's cube corresponds in this two-dimensional representation to twisting the first or third square layer. Twists of the other four sides of the cube look somewhat more complicated in the two-dimensional representation, since they cause cubies to move between layers, and they cause some colors that start out facing up or down on the cube to face toward the sides and vice-versa. The reader might enjoy working out how these twists would look to a two-dimensional person using this representation of the Rubik's cube.

By analogy, we can imagine the $3\times3\times3\times3$ tesseract as three $3\times3\times3$ cubes that are stacked "up" in the fourth dimension. All three cubes have the same six colors assigned to their faces, and in addition there are two more colors assigned to the interiors of all of the cubies in the first and third cubes. (We will continue to refer to the 81 small cubes in this representation as "cubies," although each actually represents one of the 81 small tesseracts that make up the Rubik's tesseract.) We can picture it as three Rubik's cubes that are identical, except that the first and third are made out of colored plastic instead of the usual black plastic. This is illustrated in FIGURE 3; the colored stickers on the sides of the cubies are shown in this figure as being smaller than the stickers on the sides of a real Rubik's cube, to allow the colors of the plastic of the first and third cubes to show through around the edges. The computer simulation of the tesseract mentioned above displays a picture similar to FIGURE 3 on the screen of the computer.

Of course, the computer simulation also maintains internally a mathematical description of the Rubik's tesseract in which the positions of the 81 cubies are represented by assigning them coordinates in $\mathbf{R}^4$. Although this representation will not be used in the discussion below, the reader may be interested in a brief description of it. The coordinate system is set up in such a way that the centers of the 81 cubies are the points of $\mathbf{R}^4$ all of whose coordinates are -1, 0, or 1. Each cubie has an exposed side—and hence a color—facing in the direction of each dimension for which the corresponding coordinate of its center is nonzero. The twists of the "faces" of the Rubik's tesseract, which are described geometrically below, are computed mathematically by the computer by applying appropriate rotations in $\mathbf{R}^4$ to certain subsets of the tesseract.

The three-dimensional Rubik's cube has six square faces, each with a different color assigned to it; the tesseract has eight cubical "faces." The first and third cubes represent two of the faces of the tesseract, and the colors in their interiors face in opposite directions in the fourth dimension when the cubes are stacked "up" to form a tesseract. The gray stickers on the tops of all three cubes identify the top layers of the three cubes as making up another face of the tesseract. Similarly, the fronts, backs, bottoms, and left and right sides of the three cubes make up the other five faces.

The Rubik's cube contains three kinds of cubies: corner cubies, which have three colors attached to them; edge cubies, with two colors; and cubies in the centers of the faces, which have only one color. (We are ignoring the cubie in the center of the cube, which has no color attached to it and plays no role in the puzzle. In fact, readers who have taken their Rubik's cubes apart know that there actually is no cubie in the center.) Note that in some cases cubies of the same type look quite different in the two-dimensional representation of the cube described above. For example, the corners of the middle square layer and the edges of the first and third layers all represent edge cubies.

The pieces of the tesseract fall into four categories, which can be identified by the number of colors attached to them. For example, the corners of the middle cube in FIGURE 3 and the edges of the first and third cubes all have three colors attached to them, and therefore all belong to the category of three color cubies; we will call these *3C cubies*. Note that again we are ignoring the cubie in the center of the center cube in FIGURE 3, but the centers of the first and third cubes have a color assigned to their interiors, so they are 1C cubies. The reader can check by studying FIGURE 3 that there are a total of 8 1C cubies, 24 2C cubies, 32 3C cubies, and 16 4C cubies.

The tesseract can be scrambled by twisting any of its eight faces. The twists that are easiest to understand in FIGURE 3 involve rotating the first and third cubes, which we have already seen represent two of the eight faces. Since the faces are cubical rather than square, they can be rotated in many different directions. For example, we

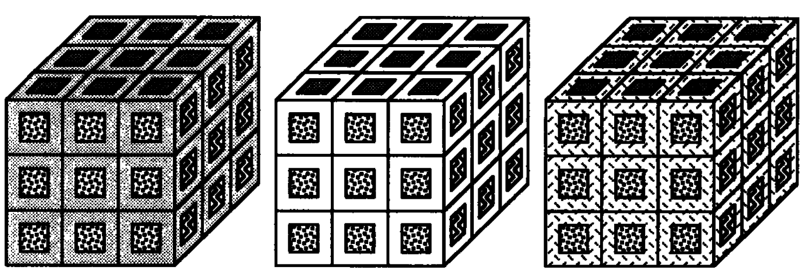

FIGURE 3

could rotate the first cube 90° around a vertical axis to bring the zigzag color on its right side to the front. Note that this rotation can be thought of as simultaneously twisting the lower, middle, and upper layers of the first cube in the same direction. Similarly, we can imagine slicing this cube into either front, middle, and back layers or left, middle, and right layers, and there are other rotations of the cube that will result in these layers being twisted simultaneously 90° in the same direction. Other rotations are also possible, such as rotations around diagonal axes, but note that they can all be accomplished by composing the three types of 90° rotations.

Now let's consider the face consisting of the right sides of all three cubes. These sides form another cubical face when the cubes are stacked "up" in the fourth dimension, but our inability to visualize this stacking makes some of the possible rotations of this face difficult to understand. However, there is one rotation of this face that is easy to understand. The remarks in the last paragraph should make it clear that there is a way of rotating this face that has the effect of simultaneously twisting the right sides of all three cubes in the same direction. Similarly, we can twist any side of the three cubes in any direction, as long as we perform the same twist simultaneously on all three cubes. There are other rotations of faces of the tesseract that are harder to understand because they move some cubies from one cube to another, and they cause the colors assigned to the interiors of some cubies to move to their surfaces, and vice-versa. Readers who have worked out how twists of the sides of the Rubik's cube would look to a two-dimensional person should be able to figure out the effects of these more complicated rotations. Fortunately, it is possible to analyze the mathematics of the tesseract without understanding in detail what these rotations do.

The key to unscrambling both the cube and the tesseract is to find sequences of twists whose net effect is to perform some simple, useful operation on the cube or tesseract. Sequences of twists are called *processes*, and the sets of all processes on the cube and tesseract form groups under composition, when processes that have the same effect are identified. It should be clear from the last paragraph that any Rubik's cube process can be applied simultaneously to all three cubes of the tesseract. However, there is a simple trick that makes it possible to perform some cube processes on *only one* cube of the tesseract, without affecting the other two cubes at all. Readers who want to solve the tesseract without getting any hints might want to try to find this trick for themselves before reading the next paragraph.

Consider the following three-step tesseract process. First rotate the first cube so that the gray stickers that start out on top end up facing to the right, then twist the right sides of all three cubes 90° clockwise, and finally undo the first rotation to return the gray side of the first cube to the top. The net effect of this process is to twist the top of the first cube and the right sides of the other two cubes 90° clockwise. Similar processes can be used to twist simultaneously the right sides of the last two cubes and any side of the first cube, and by composing such processes we can perform any cube process on the first cube, simultaneously twisting the right sides of the other two cubes some number of times. Let us define the *total twist* of a cube process to be the total number of 90° clockwise twists in the process. (A 90° counterclockwise twist can be treated as three 90° clockwise twists.) Then it should be clear that if this procedure is used on a cube process whose total twist is a multiple of 4, then the result will be a tesseract process whose net effect is to perform this cube process on the first cube of the tesseract, leaving the other two cubes unchanged. A similar procedure can be used to apply any cube process whose total twist is a multiple of 4 to any of the three cubes of the tesseract, without changing the other two cubes.

In fact, *any* cube process can be applied to the middle cube of the tesseract. The reader is invited to try proving this by finding a four-move process that twists a side of the middle cube 90°, leaving the rest of the tesseract fixed. By a *move* here we mean any reorientation of a face of the tesseract, including 180° rotations and rotations around diagonal axes. (Here's one way to approach this problem: First find a four-move process that twists the middle layer of the first cube 90°, leaving the rest of the tesseract fixed. Then figure out why this is essentially the same as the original problem.) Cube processes whose total twist is even can also be applied to the first and third cubes of the tesseract, but I know of no easy proof of this. (A somewhat indirect proof can be constructed using the analysis of possible positions of the tesseract, which is presented below. According to this analysis, the result of applying a cube process with even total twist to the first or third cube is a possible position.) The shortest tesseract process I have found that twists one face of the first cube 180° and fixes the rest of the tesseract is 35 moves long.

Fortunately, many useful cube processes have a total twist that is a multiple of 4. Some of them are commutators—elements of the process group of the form $aba^{-1}b^{-1}$—and clearly the total twist of a commutator is always a multiple of 4. For both the cube and the tesseract, it is useful to consider two kinds of processes: those that change the locations of cubies, and those that leave the locations of all cubies fixed but change the orientations of some cubies. In the first category, there are well-known commutator cube processes that change the locations of just three corner cubies or three edge cubies, leaving the locations and orientations of all other cubies fixed (see [1] and [3]); we will use the language of permutation groups and call these *corner 3-cycles* and *edge 3-cycles*. In the second category, there are commutator processes that change the orientations of two edge cubies and others that twist two corner cubies 120° in opposite directions, leaving the rest of the cube unchanged. All of these processes have a total twist that is a multiple of 4, so the technique described above can be applied to them, resulting in tesseract processes that perform 3-cycles of corners or edges of any of the three cubes, processes that flip two edge cubies of any cube, and processes that twist two corner cubies of any cube in opposite directions. It is not hard to modify these processes slightly to find processes that perform arbitrary 3-cycles of 2C, 3C, or 4C cubies, and processes that change the orientations of pairs of 2C, 3C, or 4C cubies from different cubes. At this point it may seem that the tesseract is not very different from the cube, but there are a few surprises still to come.

If we ignore the orientations of the cubies, it is not hard now to analyze which permutations of cubies can be achieved by rotating the sides of the tesseract. Consider again the rotation of the first cube 90° around a vertical axis, bringing the zigzag color on its right side to the front. Looking at the effect of this rotation on the different categories of cubies, we see that it results in a 4-cycle of 2C cubies, three disjoint 4-cycles of 3C cubies, and two disjoint 4-cycles of 4C cubies. Clearly the same would be true of any 90° rotation of any face, so each such rotation results in an odd permutation of 2C and 3C cubies and an even permutation of 4C cubies. Note that 1C cubies are not affected by any rotations, so we can ignore them from now on. Since every process can be written as a composition of 90° rotations of faces, every process must cause an even permutation of 4C cubies, and permutations of the 2C and 3C cubies that are either both even or both odd.

Since every even permutation can be written as a composition of 3-cycles, we can use the tesseract 3-cycle processes derived above to achieve any even permutations of the 2C, 3C, and 4C cubies. To reach a configuration in which the permutations of the 2C and 3C cubies are both odd, first do a 90° rotation of any face. Now the permutations of the 2C and 3C cubies required to reach the desired configuration are even, and can therefore be achieved as before. Thus the possible permutations of cubies are precisely those that consist of an even permutation of 4C cubies and permutations of 2C and 3C cubies that have the same parity.

To analyze how the orientations of cubies can be changed by rotating sides of the tesseract, it will be useful to introduce some notation. First, let us fix a numbering of the dimensions of the tesseract. We will let dimension number 1 be the front-to-back dimension, dimension 2 the left-to-right dimension, dimension 3 the top-to-bottom dimension, and of course dimension 4 will be the fourth dimension, represented in FIGURE 3 by the assignment of cubies to different cubes. We define the *dimension number* of any color to be the number of the dimension in which that color faces when the tesseract is unscrambled. For example, the gray stickers on all three cubes face up when the tesseract is unscrambled, so the dimension number of gray is 3. Remember that the colors in the interiors of the first and third cubes face in the fourth dimension, so their dimension number is 4.

Finally, in any position of the tesseract we assign to each cubie an *orientation vector* $\mathbf{c} = (c_1, c_2, c_3, c_4)$, where $c_i =$ the dimension number of the color that is facing in dimension i on this cubie. If there is no color facing in dimension i, we let $c_i = 0$. Of course, when the tesseract is unscrambled we always have either $c_i = i$ or $c_i = 0$, but when the tesseract is scrambled colors sometimes face in directions other than their original directions, so we may have $0 \neq c_i \neq i$. Note that the dimension numbers of colors are fixed, but the orientation vectors of cubies can change when sides of the tesseract are twisted.

For example, consider the top right edge cubie of the first cube in FIGURE 3. It has no color facing forwards or backwards, so the first coordinate of its orientation vector is 0, but it does have colors facing in the other three dimensions. Thus its orientation vector is $(0,2,3,4)$. Now suppose we rotate the first cube 90° around a vertical axis, bringing this cubie to the front. Then the zigzag color on its right side will move to the front, and in this new position there will be no color on this cubie facing toward the left or right. Since the dimension number of the zigzag color is 2, the orientation vector of this cubie after the twist will be $(2,0,3,4)$. Note that the first two coordinates of the orientation vector have been exchanged by this twist. In fact, the reader should be able to verify that this twist causes the first two coordinates of the orientation vectors of all the cubies it affects to be exchanged. The same is true of other 90° rotations of faces, except that different rotations may cause different pairs of coordinates to be exchanged.

Using this notation, we can now analyze the possible orientations of cubies when the tesseract is scrambled. Let us first consider the orientations of the 2C cubies. Each of these cubies has only two nonzero entries in its orientation vector. We will call a 2C cubie *sane* if the nonzero entries in its orientation vector appear in increasing order, and *flipped* otherwise. Of course, all 2C cubies are sane before the tesseract is scrambled.

When the first cube is rotated 90° around a vertical axis, the four 2C cubies in its middle layer are affected. All of them have an interior color that faces in dimension 4 (and thus the fourth coordinates of their orientation vectors are nonzero), and in addition two have colors facing in dimension 1 and two have colors facing in dimension 2. The twist exchanges the first two coordinates of the orientation vectors of all four cubies. In general, for any 90° rotation of a face of the tesseract there will be only four 2C cubies affected, and there will be distinct numbers i, j, and k such that the twist exchanges coordinates i and j of the orientation vectors of the cubies, all four of the cubies have nonzero kth coordinates in their orientation vectors, and in addition two have nonzero ith coordinates and two have nonzero jth coordinates. Clearly if k is between i and j, then all four cubies will have their sanities switched by this twist, and otherwise their sanities will be unchanged. It follows that the number of flipped 2C cubies will be even in all possible positions of the tesseract. Thus if we are told the locations and orientations of all 2C cubies except one in some scrambled position of the tesseract, we can deduce the orientation of the last 2C cubie. Since we have already seen that there are processes that flip the 2C cubies two at a time, this is the only restriction on the possible orientations of the 2C cubies.

Consider now any 3C cubie. Before the tesseract is scrambled it has an orientation vector $\mathbf{c} = (c_1, c_2, c_3, c_4)$ that has three nonzero entries. After some process has been executed, it will have a new orientation vector $\mathbf{c}'$ that is a permutation of $\mathbf{c}$. We will call the orientation of this cubie *even* or *odd*, according to whether this permutation is even or odd. Clearly the orientations of all 3C cubies are even before the tesseract is scrambled, and a 90° twist of a face affects 12 3C cubies, transposing two coordinates of their orientation vectors and therefore changing the parities of their orientations. Thus in any position of the tesseract there must be an even number of 3C cubies whose orientations are odd. Note that, unlike the restriction given above on the orientations of 2C cubies, this does not give us enough information to determine the orientation of a 3C cubie, even if we know the locations and orientations of all other cubies. For example, consider a process that leaves the locations and orientations of all cubies fixed, except perhaps for one 3C cubie. We know the orientation of this cubie must be even, but the three colors attached to this cubie can be permuted in $3! = 6$ ways, and 3 of these are even permutations. Thus, either there are other restrictions on the possible orientations of the 3C cubies, or there must be processes which change the orientation of a single 3C cubie, leaving the rest of the tesseract fixed. Experience with the Rubik's cube suggests that the first of these possibilities is the most likely, but in fact the second is correct.

To see how to construct a process that changes the orientation of a single 3C cubie, recall that we already know how to flip two edge cubies on the first or third cube. By modifying these processes we can in fact exchange any two colors on any two 3C cubies, fixing the rest of the tesseract. Now let C_1, C_2, and C_3 be any three 3C cubies. Let p be a process that exchanges the first two colors on C_1 and any two colors on C_2, and let q be a process that exchanges the last two colors on C_1 and any two colors on C_3. We let the reader verify that the commutator $pqp^{-1}q^{-1} = (pq)^2$ performs a 3-cycle on the colors attached to C_1, and has no net effect on C_2 or C_3. (The reader might enjoy trying to find a more efficient process that produces the same result. My shortest solution takes 20 moves.) Thus any even permutation can be performed on the colors attached to any 3C cubie. Combining this with the fact that any two colors on any two 3C cubies can be exchanged, we can conclude that the only restriction on the orientations of 3C cubies is the one we have already stated, that the number of 3C cubies with odd orientations must be even.

Finally, consider the 4C cubies. The four colors attached to any 4C cubie can be permuted in $4! = 24$ ways, but half of these can be ruled out immediately by a parity argument. To prove this, we set up a four-dimensional coordinate system in which the coordinates of the centers of the 4C cubies are $(\pm 1, \pm 1, \pm 1, \pm 1)$. We define the *sign* of the location of a 4C cubie to be the product of the coordinates of its center. Thus, the location is positive if there are an even number of -1's in the coordinates, and negative if there are an odd number of -1's. As before, we also call the orientation of a 4C cubie even or odd depending on how its orientation vector has been permuted. The reader can now check that a 90° rotation of a face affects eight 4C cubies, changing both the sign of the location and the parity of the orientation of each. Therefore if a cubie's location has the same sign as it had before the tesseract was scrambled, then its orientation must be even, and if not, its orientation must be odd. In particular, any process that does not change the locations of any 4C cubies can only perform even permutations on their orientation vectors.

This still leaves us with 12 possible orientations for each 4C cubie. Using processes we have already discussed we can simultaneously put 15 of the 16 4C cubies in any of these 12 orientations, as follows. We have already found processes that twist two corners of the first or third cube 120° in opposite directions. Variations on these processes will allow us to perform any 3-cycle on the orientation vector of any 4C cubie, simultaneously performing a 3-cycle on the orientation vector of some other 4C cubie as well. Combining these processes we can therefore perform any combination of even permutations on the orientation vectors of all the 4C cubies except one. We must still determine the possible orientations for this last 4C cubie.

It will turn out that there are fewer than 12 possible orientations for the last 4C cubie, but based on our discussion of 3C cubies the reader can probably guess that there will be more than one possibility. In fact, the proof of this is very similar to the proof for 3C cubies. Let C_1, C_2, and C_3 be three 4C cubies. Let p be a process that performs a 3-cycle on the first three coordinates of the orientation vector of C_1, simultaneously permuting the orientation vector of C_2, and let q be a process that performs a similar 3-cycle on the last three coordinates of the orientation vector of C_1, simultaneously permuting the orientation vector of C_3. Then the commutator $pqp^{-1}q^{-1}$ transposes the first and last coordinates, and also the second and third coordinates, of the orientation vector of C_1, leaving the rest of the tesseract fixed. Similarly we can transpose any two disjoint pairs of coordinates of the orientation vector of any 4C cubie. (Again, there are more efficient ways of accomplishing this. My shortest process takes 16 moves.)

Let us say that two orientation vectors are *similar* if either they are equal, or we can transpose two disjoint pairs of coordinates of one to get the other. It is easy to check that this is an equivalence relation, and each equivalence class has four elements. To complete the analysis of the possible orientations of 4C cubies, it will be useful to choose representatives of these equivalence classes, which we do as follows. We say that an orientation vector $\mathbf{c}=(c_1,c_2,c_3,c_4)$ is *normal* if $c_4=4$, and we define the *normal form* of $\mathbf{c}$ to be the unique vector $\bar{\mathbf{c}}=(\bar{c}_1,\bar{c}_2,\bar{c}_3,\bar{c}_4)$ such that $\bar{\mathbf{c}}$ is similar to $\mathbf{c}$ and $\bar{\mathbf{c}}$ is normal. Since we have shown that the orientation of any 4C cubie can be changed to any other similar orientation without changing the rest of the tesseract, it suffices now to determine the possible normal forms for the orientation vector of the last 4C cubie.

For future use we observe that if $\mathbf{c}$ and $\mathbf{c}'$ are similar orientation vectors and we transpose the same pair of coordinates of both, then the resulting vectors are still similar. Therefore, if we transpose the first and second, first and third, or second and third coordinates of $\mathbf{c}$, the normal form of the resulting vector can be found by performing the same transposition of $\bar{\mathbf{c}}$. Transposing the third and fourth coordinates of $\mathbf{c}$ yields the same result as transposing the first and second coordinates of the similar vector (c_2,c_1,c_4,c_3), and therefore the normal form of the resulting vector can be found by transposing the first and second coordinates of $\bar{\mathbf{c}}$. Similarly it can be shown that any transposition of two coordinates of $\mathbf{c}$ causes some two of the first three coordinates of the normal form of $\mathbf{c}$ to be transposed.

For an orientation vector $\mathbf{c}$ that is normal we define the *twist* of $\mathbf{c}$ to be the value of i for which $c_i=3$. If $\mathbf{c}$ is not normal, we define the twist of $\mathbf{c}$ to be the twist of its normal form—i.e., the unique i for which $\bar{c}_i=3$. The twist of a cubie is the twist of its orientation vector. We are now ready to state the last restriction on the orientations of 4C cubies. We claim that the sum of the twists of the cubies in positive locations and the sum of the twists of cubies in negative locations are always congruent mod 3. Equivalently, if we define the *signed twist* of a cubie to be equal to the twist of the cubie times the sign of its location, then the sum of the signed twists of all cubies is always a multiple of 3.

Clearly, before the tesseract is scrambled, all 4C cubies have orientation vector $(1, 2, 3, 4)$, which is normal, so all 4C cubies have a twist of 3. Thus the sum of the signed twists of all cubies is 0. Now consider any 90° rotation of a face of the tesseract. Recall that such a rotation transposes the same two coordinates of the orientation vectors of eight 4C cubies, simultaneously reversing the signs of their locations. As we have already observed, the effect of this operation on the normal forms of the orientation vectors of these cubies will be to cause some two of the first three coordinates of these normal forms to be transposed. Now consider the effect of this transposition on the twists of these cubies. The reader can easily check that if the twist of some normal orientation vector is t, then transposing the first two coordinates of this vector gives a normal orientation vector whose twist is congruent to $-t \bmod 3$. Similarly, transposing the first and third coordinates results in a twist congruent to $1-t \bmod 3$, and transposing the second and third gives a twist congruent to $2-t \bmod 3$. Thus rotating any face 90° will perform one of these three transformations on the twists of the eight cubies affected by the rotation.

Finally, we consider the effect of a 90° rotation on the signed twists of the 4C cubies affected. According to the last paragraph, for each such rotation there is a constant k such that each affected cubie with a twist of t before the rotation has a twist congruent to $k-t \bmod 3$ after rotation. Since the sign of the location of each of these cubies is reversed by the rotation, a cubie in a positive location whose twist is t has its signed twist changed from t to $t-k \bmod 3$, while if the location is negative then the signed twist changes from $-t$ to $k-t \bmod 3$. Since half of the signed twists are decreased by k and half are increased by $k \bmod 3$, the sum of the signed twists is unchanged mod 3. This proves our claim that the sum of the signed twists of the 4C cubies is a multiple of 3 in all possible positions of the tesseract.

If we are told the locations and orientations of 15 of the 16 4C cubies, we can now determine what orientations are possible for the last 4C cubie. From the restrictions derived above we can determine both the parity and the twist of the orientation. If the orientation vector $\mathbf{c}$ of this cubie is normal then we know $c_4 = 4$, and the twist of the cubie tells us the value of i for which $c_i = 3$. The values of the other two entries of the orientation vector are then determined by the parity of the orientation. Thus there is only one possible normal orientation vector for the last 4C cubie. Since any orientation vector is possible if and only if its normal form is, the possible orientations are just the four which are similar to this normal orientation.

We have now given a complete analysis of the possible scrambled positions of the tesseract. Using this analysis we can compute the number of such positions—i.e., the order of the process group. Considering first the locations of the cubies and ignoring their orientations, the 24 2C cubies, 32 3C cubies, and 16 4C cubies can be permuted in $24! \times 32! \times 16!$ ways. But only even permutations of the 4C cubies are possible, and the parities of the permutations of the 2C and 3C cubies must be the same. Thus the number of these permutations that can be achieved by rotating the sides of the tesseract is $(24! \times 32!)/2 \times 16!/2$. For each of these permutations, 23 of the 2C cubies can have either of two orientations, with the orientation of the last 2C cubie then being determined. Thirty-one of the 3C cubies can have any of six orientations, with the orientation of the last 3C cubie being restricted by its parity to only three possibilities, and 15 of the 4C cubies have 12 possible orientations, with the last having only four possibilities. Thus the total number of positions of the tesseract is:

$$\begin{aligned}&(24!\times 32!)/2\times 16!/2\times 2^{23}\times 6^{31}\times 3\times 12^{15}\times 4\\ &=1{,}756{,}772{,}880{,}709{,}135{,}843{,}168{,}526{,}079{,}081{,}025{,}059{,}614,\\ &\quad 484{,}630{,}149{,}557{,}651{,}477{,}156{,}021{,}733{,}236{,}798{,}970{,}168,\\ &\quad 550{,}600{,}274{,}887{,}650{,}082{,}354{,}207{,}129{,}600{,}000{,}000{,}000,\\ &\quad 000\\ &\cong 1.76\times 10^{120}.\end{aligned}$$

For comparison, we note that the number of positions for the Rubik's cube is a measly 4.33×10^{19}.

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The Catalan Numbers and Pi

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The Catalan numbers $\binom{2n-2}{n-1}/n$, $n = 1, 2, \dots,$ arise naturally in many problems of *discrete* mathematics. For a vigorous discussion of some of these problems see the recent article of R. B. Eggleton and R. K. Guy [3]. The universal constant π is truly ubiquitous throughout mathematics and the empirical sciences. However, the constant occurs most frequently in elementary calculus, which is the cornerstone of *continuous* mathematics. The following series representation of $1/\pi$ relates the Catalan numbers and π in a curious manner.

$$\frac{1}{\pi} = \frac{3}{16} + \frac{9}{4} \sum_{k=1}^{\infty} \left\{ \frac{\binom{2k-2}{k-1}}{k} \right\}^2 \frac{4k^2 - 1}{2^{4k}(k+1)^2}. \tag{1}$$

In [4, pp. 36–38] Ramanujan presented 17 series representations of $1/\pi$, and within the confines of elliptic function theory proved three of these. Apparently, Ramanujan had very little interest in combinatorics, and accordingly made no attempt to interpret the terms of the series in terms of interesting combinatorial objects. However, several of his series have terms that involve the central binomial coefficients $\binom{2n}{n}$, $n = 0, 1 \dots$. Unlike the series representations of Ramanujan our representation (1) requires no advanced machinery for its justification. In fact, all of the tools can be found in any good elementary calculus textbook.